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Local convergence for random permutations and the case of uniform ρ-avoiding permutations with $|\rho| = 3$

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For large combinatorial structures, two main notions of convergence can be usually defined: scaling limits and local limits. Informally scaling limits consist in studying the objects from a global point of view (after a rescaling of the distances between points of the objects), while for local limits we study the objects in a neighborhood around a marked point (without rescaling distances). In particular for graphs, both notions are well-studied and well-understood. For permutations only a notion of scaling limits, called permutons, has been recently introduced (see [5]). The convergence in the sense of permutons has also been characterized by frequencies of pattern occurrences (see [2]).

Our main results can be divided into two different parts:

1. We set up a new notion of local convergence for permutations and we prove a characterization in terms of proportions of *consecutive* pattern occurrences. We are also able to characterize random limiting objects introducing a "shift-invariant" property.

2. We show examples of local convergence in the framework of random pattern-avoiding permutations: we describe the asymptotics in n of the number of consecutive occurrences of any fixed pattern π in a uniform ρ-avoiding permutation of size n, for $|\rho| = 3$. For this last result we use bijections between ρ-avoiding permutations and ordered rooted trees and singularity analysis.

1 Local convergence for permutations

This section is inspired by local convergence for trees (see *e.g.* [1]) and Benjamini-Schramm convergence for random graphs (see [3]). In the context of local convergence we need to look at permutations with a marked entry, called root. We denote with $\mathcal{S}^n$ the set of permutations of size n and with $\mathcal{S}$ the set of permutations of finite size.

Definition 1. A *finite rooted permutation* is a pair (σ, i), where $\sigma \in \mathcal{S}^n$ and $i \in [n]$.

To a rooted permutation (σ, i), we associate (as shown in the left hand-side of Fig. 1) the pair $(A_{\sigma,i}, \preccurlyeq_{\sigma,i})$, where $A_{\sigma,i} := [-i+1, |\sigma| - i]$ is a finite interval of integers containing 0 and $\preccurlyeq_{\sigma,i}$ is a total order on $A_{\sigma,i}$, defined for all $\ell, j \in A_{\sigma,i}$ by $\ell \preccurlyeq_{\sigma,i} j$ if $\sigma_{\ell+i} \leq \sigma_{j+i}$.

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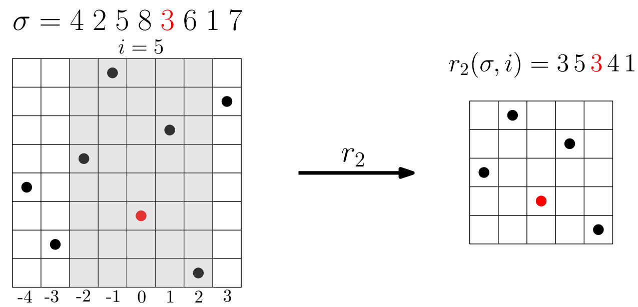
Given the diagram of a rooted permutation (σ, i), the corresponding total order $(A_{\sigma,i}, \preccurlyeq_{\sigma,i})$ is simply obtained by shifting the origin of the x-axis to the column containing the root of the permutation (the new indices are reported under the columns of the diagram) and then reading the diagram of the permutation from bottom to top, gradually recording, for each element, its position according to the new x-axis.



$2 \leq_{\sigma,i} -3 \leq_{\sigma,i} 0 \leq_{\sigma,i} -4 \leq_{\sigma,i} -2 \leq_{\sigma,i} 1 \leq_{\sigma,i} 3 \leq_{\sigma,i} -1$ $\quad$ $2 \leq_{\sigma,i} 0 \leq_{\sigma,i} -2 \leq_{\sigma,i} 1 \leq_{\sigma,i} -1$

Figure 1: Restriction function for rooted permutations.

Since the map from the space of finite rooted permutations to the space of total orders on finite integer intervals containing zero is a bijection, we identify every rooted permutation (σ, i) with the total order $(A_{\sigma,i}, \preccurlyeq_{\sigma,i})$.

Thanks to this identification, the following definition of infinite rooted permutation is natural.

Definition 2. We call *infinite rooted permutation* a pair $(A, \preccurlyeq)$ where A is an infinite interval of integers containing 0 and $\preccurlyeq$ is a total order on A.

The next step is to define a notion of local convergence in the space of rooted (finite or infinite) permutations denoted by $\tilde{\mathcal{S}}_\bullet$. In order to do that we introduce a notion of neighborhoods around the root which can be thought as a "vertical strip" around the root of the permutation (see Fig. 1). Formally, for all $h \in \mathbb{N}$, we define the *restriction function around the root* as

$$\begin{aligned} r_h \colon \quad \tilde{\mathcal{S}}_\bullet &\longrightarrow \quad \tilde{\mathcal{S}}_\bullet \\ (A, \preccurlyeq) &\mapsto \big(A \cap [-h, h], \preccurlyeq\big)\ . \end{aligned}$$

The restriction r_h of the diagram around the root coincide with the diagram of the rooted permutation induced by the pattern $\text{pat}_{[a,b]}(\sigma)$ where $a = \max\{i - h, 1\}$ and $b = \min\{i + h, |\sigma|\}$.

Example 3. We refer to Fig. 1. On the top of the picture we see the restriction $r_2(\sigma, i)$ from the diagram point of view and on the bottom from the total order point of view.

Definition 4. We say that a sequence $(A_n, \preccurlyeq_n)_{n \in \mathbb{N}}$ of rooted permutations in $\tilde{\mathcal{S}}_\bullet$ is *locally convergent* to an element $(A, \preccurlyeq) \in \tilde{\mathcal{S}}_\bullet$, if for all $h > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$r_h(A_n, \preccurlyeq_n) = r_h(A, \preccurlyeq).$$

This topology is metrizable by a local distance d and we prove that the space $(\tilde{\mathcal{S}}_\bullet, d)$ is a compact Polish space.

We have defined a notion of local convergence for rooted permutations and we want to extend this notion to study sequences of (unrooted) permutations. We can see a fixed permutation σ as a rooted object only after a root i has been chosen. A natural way to choose a root is to make the choice at random, and uniformly among the indices of σ. In this way, a fixed permutation σ naturally identifies a random variable $(\sigma, \boldsymbol{i})$ that takes values in the set of finite rooted permutations (we denote random quantities using **bold** characters).

Definition 5. Given a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{S}$, we say that $(\sigma_n)_{n\in\mathbb{N}}$ *Benjamini- Schramm converges* to a random (possibly infinite) rooted permutation $(\boldsymbol{A}, \boldsymbol{\preccurlyeq})$, if the sequence $(\sigma_n, \boldsymbol{i}_n)_{n\in\mathbb{N}}$, where $\boldsymbol{i}_n$ is a uniform index in $[1, |\sigma_n|]$, converges in distribution to $(\boldsymbol{A}, \boldsymbol{\preccurlyeq})$ with respect to the local distance d.

We prove the following characterization in terms of proportions of consecutive pattern occurrences. For any pattern π of size k, and any permutation σ of size n, we denote by

$$\widetilde{c\text{-}occ}(\pi, \sigma) = \frac{\text{number of consecutive occurrences of } \pi \text{ in } \sigma}{n},$$

the proportion of consecutive occurrences of π in σ.

Theorem 6. *For any $n \in \mathbb{N}$, let σ^n be a permutation of size n. Then the Benjamini-Schramm convergence for the sequence $(\sigma^n)_{n\in\mathbb{N}}$ is equivalent to the existence of an infinite vector of non-negative real numbers $(\Delta_\pi)_{\pi\in\mathcal{S}}$ such that, for all patterns $\pi \in \mathcal{S}$,*

$$\widetilde{c\text{-}occ}(\pi, \sigma^n) \to \Delta_\pi.$$

We then extend, in two non-equivalent ways, the above notion of Benjamini-Schramm convergence to sequences of *random* permutations $(\boldsymbol{\sigma}^n)_{n\in\mathbb{N}}$ introducing the *annealed* and the *quenched* version of the Benjamini-Schramm convergence. We obtain the following two characterizations:

Theorem 7. *For any $n \in \mathbb{N}$, let $\boldsymbol{\sigma}^n$ be a random permutation of size n. Then*

(a) The annealed version of the Benjamini-Schramm convergence of $(\boldsymbol{\sigma}^n)_{n\in\mathbb{N}}$ is equivalent to the existence of an infinite vector of non-negative real numbers $(\Delta_\pi)_{\pi\in\mathcal{S}}$ such that for all patterns $\pi \in \mathcal{S}$,

$$\mathbb{E}[\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n)] \to \Delta_\pi.$$

(b) The quenched version of the Benjamini-Schramm convergence of $(\boldsymbol{\sigma}^n)_{n\in\mathbb{N}}$ is equivalent to the existence of an infinite vector of non-negative real random variables $(\boldsymbol{\Lambda}_\pi)_{\pi\in\mathcal{S}}$ such that

$$\left(\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n)\right)_{\pi\in\mathcal{S}} \stackrel{(d)}{\to} (\boldsymbol{\Lambda}_\pi)_{\pi\in\mathcal{S}},$$

w.r.t. the product topology (where $\stackrel{(d)}{\to}$ indicates the convergence in distribution).

Obviously, the quenched version implies the annealed version.

Finally we are also able to characterize random limiting objects for the annealed version of the Benjamini-Schramm convergence introducing a "shift-invariant" property (corresponding to the well-known notion of unimodularity for random graphs).

2 Local convergence for uniform ρ-avoiding permutations with $|\rho| = 3$

We show the relevance of this new notion of convergence proving the local convergence for random ρ-avoiding permutations and characterizing their limits.

Theorem 8. *Let $\rho \in \mathcal{S}^3$ and for any $n \in \mathbb{N}$, let $\boldsymbol{\sigma}^n$ be a uniform random ρ-avoiding permutation. Then we have the following convergence in probability,*

$$\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n) \xrightarrow{P} P_\rho(\pi), \quad \textit{for all} \quad \pi \in \text{Av}(\rho), \tag{2.1}$$

where for all $m \in \mathbb{N}$, $\big(P_\rho(\pi)\big)_{\pi \in \text{Av}^m(\rho)}$ is a probability distribution on $\text{Av}^m(\rho)$ given below.

An interesting aspect of the theorem is the condensation phenomenon, indeed the limits of the random sequences $\big(\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n)\big)_{n \in \mathbb{N}}$ are deterministic, for all pattern $\pi \in \mathcal{S}$. This also implies that the convergence in probability in Equation (2.1) is equivalent to the convergence in distribution of the vector $\big(\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n)\big)_{\pi \in \mathcal{S}}$. Therefore our theorem trivially implies the characterization (b) in Theorem 7 and so prove that the sequence $(\boldsymbol{\sigma}^n)_{n \in \mathbb{N}}$ converge for the quenched version of the Benjamini-Schramm convergence. Moreover, we are able to provide a construction of the limiting random order on $\mathbb{Z}$.

We now present the main ideas of the proof of Theorem 8. First of all we note that it is enough to analyze the two cases $\rho = 321$ and $\rho = 231$. The other four cases are obtained using symmetries of the square.

We explicitly exhibit the probability distributions $\big(P_{231}(\pi)\big)_{\pi \in \text{Av}^m(231)}$ and $\big(P_{321}(\pi)\big)_{\pi \in \text{Av}^m(321)}$ that appear in the statement of the theorem. For all $m \geq 0$,

$$P_{231}(\pi) := \frac{2^{\text{LRMax}(\pi) + \text{RLMax}(\pi)}}{2^{2|\pi|}}, \quad \text{for all} \quad \pi \in \text{Av}(231),$$

where $\text{LRMax}(\pi)$ (resp. $\text{RLMax}(\pi)$) denotes the number of left-to-right maxima (resp. right-to-left maxima) in π. Moreover, for all $m \geq 0$ and for all $\pi \in \text{Av}^m(231)$,

$$P_{321}(\pi) := \begin{cases} \frac{|\pi|+1}{2^{|\pi|}} & \text{if } \pi = 12\ldots|\pi|, \\ \frac{1}{2^{|\pi|}} & \text{if } c\text{-}occ(21, \pi^{-1}) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We underline that the two limiting distributions present an important difference: the first one has full support on $\text{Av}^m(231)$, whereas the second gives positive measure only to 321-avoiding permutations whose inverse have at most one descent.

We give some hints into the proof in the case $\rho = 231$. The proof is divided into two main steps:

FIRST STEP: The goal is to prove that $\mathbb{E}[\widetilde{c\text{-}occ}(\pi, \boldsymbol{\sigma}^n)] \to P_{321}(\pi)$, for all $\pi \in \text{Av}(231)$.

We use a technique introduced by Janson [6]. Using a well-known bijection between 231-avoiding permutations and binary trees (see [4]), instead of considering a sequence of uniform 231-avoiding permutations of size n, we can consider a sequence of uniform binary trees $\boldsymbol{T}_n$ with

n vertices. The behavior of $\boldsymbol{T}_n$ can be understood considering a binary Galton-Watson tree $\boldsymbol{T}_\delta$ with offspring distribution $\eta(\delta)$, $\delta \in (0,1)$. Indeed, for some specific distribution $\eta(\delta)$, the law of a binary Galton-Watson tree $\boldsymbol{T}_\delta$ conditioned on having n vertices is equal to the law of $\boldsymbol{T}_n$. Using this result it is possible to relate $\boldsymbol{T}_\delta$ and the sequence $(\boldsymbol{T}_n)_{n\in\mathbb{N}}$ by the formula

$$\mathbb{E}\big[F(\boldsymbol{T}_\delta)\big] = \frac{1+\delta}{1-\delta}\sum_{n=1}^{+\infty}\mathbb{E}\big[F(\boldsymbol{T}_n)\big]\cdot C_n \cdot \Big(\frac{1-\delta^2}{4}\Big)^n, \quad \text{for all bounded functions } F. \tag{2.2}$$

With a recursive proof we show that

$$\mathbb{E}\big[c\text{-}occ(\pi,\boldsymbol{T}_\delta)\big] = \delta^{-1}\cdot P_{231}(\pi) + P(\delta),$$

where $P(\delta)$ is a polynomial in δ. Then applying singularity analysis to the function in Equation (2.2) (which is Δ-analytic in $z(\delta) = \frac{1-\delta^2}{4}$) we conclude that

$$\mathbb{E}\big[\widetilde{c\text{-}occ}(\pi,\boldsymbol{T}_\delta)\big] \to P_{231}(\pi), \quad \text{for all} \quad \pi \in \mathrm{Av}(231).$$

We finally conclude the proof going back to 231-avoiding permutations, working backwards the above mentioned bijection with trees"

<u>SECOND STEP</u>: The goal is to prove $\widetilde{c\text{-}occ}(\pi,\boldsymbol{\sigma}^n) \xrightarrow{P} P_{231}(\pi)$, for all $\pi \in \mathrm{Av}(231)$.

We study the second moment $\mathbb{E}[\widetilde{c\text{-}occ}(\pi,\boldsymbol{\sigma}^n)^2]$ using similar techniques as before and we obtain

$$\mathbb{E}\big[c\text{-}occ(\pi,\boldsymbol{T}_\delta)^2\big] \to P_{231}(\pi)^2, \quad \text{for all} \quad \pi \in \mathrm{Av}(231).$$

We conclude noting that

$$\mathrm{Var}\big(c\text{-}occ(\pi,\boldsymbol{T}_\delta)^2\big) \to 0, \quad \text{for all} \quad \pi \in \mathrm{Av}(231),$$

and applying the second moment method.

The proof in the case $\rho = 321$ is not presented in this abstract and uses different techniques.

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