Local convergence for permutations and local limits for uniform ρ-avoiding permutations with $|\rho| = 3$

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Part 1: Introducing local convergence for permutations

Finite rooted permutations

We view a permutations σ as a diagram, *i.e.*, a set of points of the Cartesian plane at coordinates (j, σ_j).

Finite rooted permutation

For any $n \in \mathbb{Z}_{>0}$, denoting with $\mathcal{S}^n$ the set of permutations of size n, we say that a pair (σ, i) is a *finite rooted permutation* of size n if $\sigma \in \mathcal{S}^n$ and $i \in \{1, \dots, n\}$.

To a rooted permutation (σ, i), we associate a total order $\preccurlyeq_{\sigma,i}$ on a finite interval of integers containing 0 denoted by $A_{\sigma,i}$.
The total order $(A_{\sigma,i}, \preccurlyeq_{\sigma,i})$ is simply obtained from the diagram of σ (as shown in the left-hand side of Fig. 1):

- We shift the indices of the x-axis in such a way that the column containing the root of the permutation has index zero (the new indices are reported under the columns of the diagram).
- Then we set $j \preccurlyeq_{\sigma,i} k$ if the point in column j is below the point in column k.

Infinite rooted permutations

Infinite rooted permutation

An *infinite rooted permutation* is a pair $(A, \preccurlyeq)$ where A is an infinite interval of integers, $0 \in A$ and $\preccurlyeq$ is a total order on A.

We also introduce a notion of *h-restriction* around the root.

- It can be thought of as the diagram of the pattern induced by a "vertical strip" of width $2h+1$ around the root of the permutation.
- Equivalently is the restriction of the order $(A, \preccurlyeq)$ to $A \cap [-h, h]$. This second equivalent characterization trivially applies also to infinite rooted permutations.

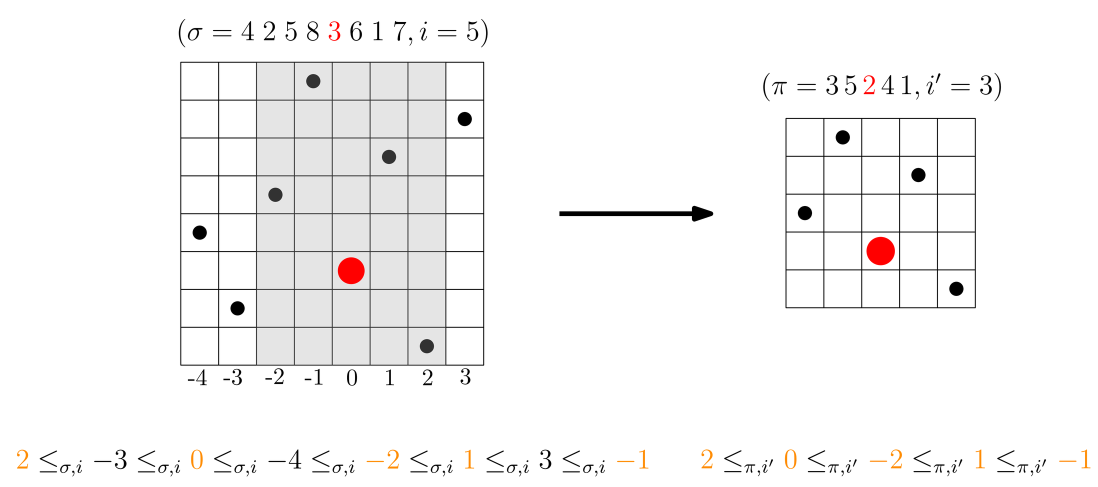


Figure 1: Two rooted permutations and the associated total orders. The second is the 2-restriction of the first.

Local convergence

Local convergence for rooted permutations

A sequence $(A_n, \preccurlyeq_n)_{n\in\mathbb{Z}_{>0}}$ of rooted permutations is *locally convergent* to a rooted permutation $(A, \preccurlyeq)$ if, for all $h \in \mathbb{Z}_{>0}$, the h-restrictions of the sequence $(A_n, \preccurlyeq_n)_{n\in\mathbb{Z}_{>0}}$ converge to the h-restriction of $(A, \preccurlyeq)$.

We extend this notion of local convergence for rooted permutations to (unrooted) permutations, rooting them at a uniformly chosen index of σ.

Local convergence for (unrooted) permutations

A sequence of permutations $(\sigma^n)_{n\in\mathbb{Z}_{>0}}$ *Benjamini–Schramm converges* to a random possibly infinite rooted permutation $(\boldsymbol{A}, \boldsymbol{\preccurlyeq})$, if the sequence $(\sigma^n, \boldsymbol{i}_n)_{n\in\mathbb{Z}_{>0}}$, where $\boldsymbol{i}_n$ is a uniform index in $[1, |\sigma^n|]$, converges in distribution to $(\boldsymbol{A}, \boldsymbol{\preccurlyeq})$ with respect to the above defined local topology.

Part 2: Main results

Characterization of local convergence

We denote the proportion of consecutive pattern occurrences of a pattern π in a permutation σ as

$$\widetilde{\text{c-occ}}(\pi, \sigma) = \frac{\#\{\text{consecutive occ. of } \pi \text{ in } \sigma\}}{n}.$$

Proposition

For any $n \in \mathbb{Z}_{>0}$, let σ^n be a permutation of size n. TFAE:

1. $(\sigma^n)_{n\in\mathbb{Z}_{>0}}$ B–S converges.
2. For all patterns $\pi \in \mathcal{S}$,
$$\widetilde{\text{c-occ}}(\pi, \sigma^n) \to \Delta_\pi.$$

We then extend the notion of B–S convergence to sequences of *random* permutations $(\boldsymbol{\sigma}^n)_{n\in\mathbb{Z}_{>0}}$ in two non equivalent versions.

Theorem

For any $n \in \mathbb{Z}_{>0}$, let $\boldsymbol{\sigma}^n$ be a random permutation of size n. Then

- The annealed version of the B–S convergence is equivalent to
$$\mathbb{E}[\widetilde{\text{c-occ}}(\pi, \boldsymbol{\sigma}^n)] \to \Delta_\pi, \quad \forall \text{ patterns } \pi \in \mathcal{S}.$$
- The quenched version of the B–S convergence is equivalent to
$$\left(\widetilde{\text{c-occ}}(\pi, \boldsymbol{\sigma}^n)\right)_{\pi\in\mathcal{S}} \xrightarrow{(d)} (\boldsymbol{\Lambda}_\pi)_{\pi\in\mathcal{S}},$$
w.r.t. the product topology.

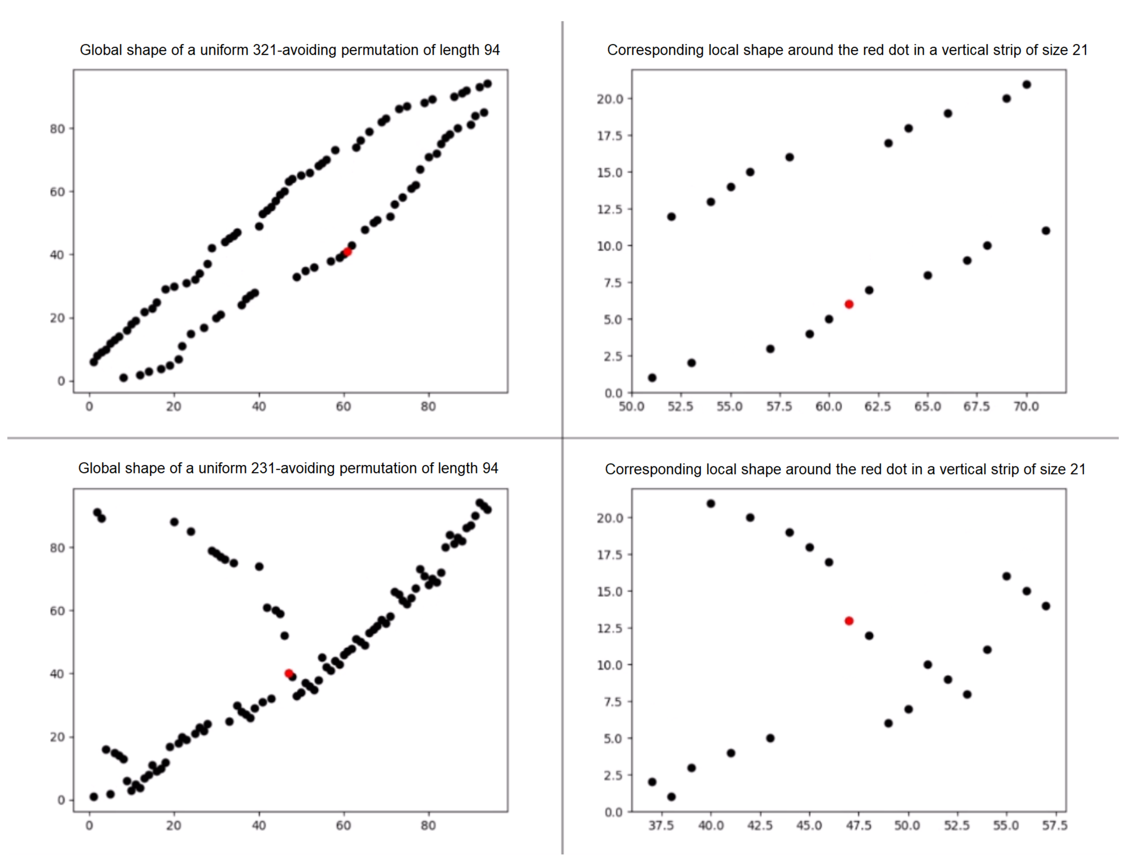
Characterization of the limiting objects

For an order $(\mathbb{Z}, \preccurlyeq)$, its *shift* $(\mathbb{Z}, \preccurlyeq')$ is defined by $i+1 \preccurlyeq' j+1$ if and only if $i \preccurlyeq j$. A random infinite permutation, or equivalently a random total order, is said to be *shift-invariant* if it has the same distribution as its shift.

Theorem

A random infinite rooted permutation is shift-invariant if and only if it is the local limit of a sequence of random permutations in the annealed Benjamini–Schramm sense.

Some simulations



ρ-avoiding permutations for $|\rho| = 3$.

We characterize the local limits of uniform permutations avoiding a pattern of size 3..

Theorem

For any $n \in \mathbb{Z}_{>0}$, let $\boldsymbol{\sigma}^n$ be a uniform random 231-avoiding permutation of size n. Then, for all $\pi \in \text{Av}(231)$,

$$\widetilde{\text{c-occ}}(\pi, \boldsymbol{\sigma}^n) \xrightarrow{P} \frac{2^{\text{LRMax}(\pi)+\text{RLMax}(\pi)}}{2^{2|\pi|}},$$

where $\text{LRMax}(\pi)$ (resp. $\text{RLMax}(\pi)$) denotes the number of left-to-right (resp. right-to-left) maxima in π.

Theorem

For any $n \in \mathbb{Z}_{>0}$, let $\boldsymbol{\sigma}^n$ be a uniform random 321-avoiding permutation of size n. Then, for all $\pi \in \text{Av}(321)$,

$$\widetilde{\text{c-occ}}(\pi, \boldsymbol{\sigma}^n) \xrightarrow{P} \begin{cases} \frac{|\pi|+1}{2^{|\pi|}} & \text{if } \pi = 12\dots|\pi|, \\ \frac{1}{2^{|\pi|}} & \text{if } \text{c-occ}(21, \pi^{-1}) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By symmetry, this covers all cases of ρ-avoiding permutations with $|\rho| = 3$. Moreover, the two theorems imply the convergence for the quenched version, *i.e.*, the stronger version, of the Benjamini–Schramm convergence.