Random combinatorial structures
Exercise sheet nb. 1

Jacopo Borga

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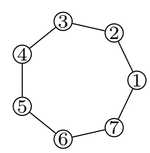
Exercise 1. By definition, a proper coloring of a graph is a coloring of its vertices such that two adjacent vertices have different colors.

- Find a coloring in three colors of the following graph:

$C_7 =$ 

By definition, the chromatic number $\chi(G)$ of a graph G is the minimum number of colors needed for a proper coloring of G.

1. What is the chromatic number of C_7 ? Same question for C_n, the obvious generalization of C_7 for a given integer n, *i.e.,* the cycle graph on n vertices.

2. Let G be a graph with maximal degree Δ (the maximal degree of a graph is the maximal number of edges incident to a vertex). Show that its chromatic number is at most $\Delta + 1$.

Exercise 2 (Fekete's lemma). A sequence (u_n) of real numbers is called *super-additive* if and only if:

$$u_{n+m} \geq u_n + u_m, \quad \text{for all} \quad n, m.$$

Show that $\lim_{n\to\infty} \frac{u_n}{n}$ exists (it may be infinite) and is equal to $\sup_{n\geq 1} \frac{u_n}{n}$.

Hint: you may first consider the case where $\frac{u_n}{n}$ has a maximal element $\frac{u_k}{k}$ and consider the Euclidean division of n by k. Then deal with the general case.

Exercise 3. Denote $\ell_n^{(k)} = \mathbb{E}[L_n^{(k)}]$ the expectation of the size of the longest common subsequence of two uniform random words of size n on a k-letter alphabet.

1. Using the previous exercise, show that $\ell_n^{(k)}$ is asymptotically equivalent to $\gamma_k n$ for some number γ_k (when n tends to infinity and k is fixed).

2. Show that for any integer k and m,

$$\gamma_{m \cdot k} \leq \gamma_k.$$

3. Recall from the lecture that $\mathbb{P}(|L_n^{(k)} - \mathbb{E}(L_n^{(k)})| \geq t_n) \leq 2\exp(-t_n^2/8n)$, when $t_n \to \infty$. Find $t_n \to \infty$ such that a.s., we have $|L_n^{(k)} - \mathbb{E}(L_n^{(k)})| \leq t_n$ for n large enough.