

Random combinatorial structures

Exercise sheet nb. 2

Jacopo Borga

March 11th, 2019

Exercise 1. The purpose of this exercise is to show that the number C_2^n of cycles of length 2 in a random uniform permutation σ^n of size n is also asymptotically Poisson, with a parameter to be determined.

Denote, for a two element set $I = \{i, j\} \subseteq \{1, \dots, n\}$,

$$\delta_I(\sigma) = \begin{cases} 1 & \text{if } \sigma(i) = j \text{ and } \sigma(j) = i; \\ 0 & \text{otherwise.} \end{cases}$$

- For distinct 2-element sets I_1, \dots, I_k compute

$$\mathbb{E}(\delta_{I_1}(\sigma^n) \cdots \delta_{I_k}(\sigma^n)).$$

Hint: beware that *distinct* 2-element sets are not necessarily *disjoint*.

- How many sets $\{I_1, \dots, I_k\}$ of k pairwise disjoint two elements subsets are there ?
- Compute the probability $P(C_2^n = 0)$.
- For a given integer r , compute $P(C_2^n = r)$ and conclude.
- (Bonus question) Fix a positive integer k . Show that the number C_k^n of cycles of length k in a random uniform permutation of size n is also asymptotically Poisson, with a parameter to be determined.

Exercise 2. Let P_n be the PGF of the number of fixed points in a uniform permutations of size n .

- Prove that

$$P_n(u) = P_{n-1}(u) + \frac{u-1}{n}(P_{n-1}(u) - P'_{n-1}(u)).$$

- Deduce, by induction the formula given in the lecture

$$P_n(u) = \sum_{k=0}^n \frac{(u-1)^k}{k!}.$$

Exercise 3 (Bonus exercise just for curious students in the spirit of ex.1 sheet 1). A coloring of the plane \mathbb{R}^2 is said to be *proper* if every pair of points at distance 1 have different colors.

- Prove that two and three colors are not enough for constructing a proper coloring of the plane,
- (Difficult question) Prove that seven colors are enough for constructing a proper coloring of the plane,
- (Difficult question) Prove that four colors are not enough for constructing a proper coloring of the plane,
- (Impossible question, solved recently) Prove that five colors are not enough for constructing a proper coloring of the plane,
- (Open problem) Are six colors enough for constructing a proper coloring of the plane?