Random combinatorial structures
Exercise sheet nb. 3

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Exercise 1. We start with some simple/classical counting problems. Find solutions that are as straightforward as possible.

Recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ counts the number of subsets of k elements in a set of n elements. (Also try to remember why.)

1. How many subsets of the set $[10] = \{1, 2, \dots, 10\}$ contain at least one odd integer?
2. In how many ways can seven people be seated in a circle if two arrangements are considered the same whenever each person has the same neighbors (not necessarily on the same side)?
3. How many permutations $\sigma : [6] \to [6]$ satisfy $\sigma(1) \neq 2$?
4. There are four men and six women. Each man marries one of the women. In how many ways can this be done?
5. Ten people split up into five groups of two each. In how many ways can this be done?
6. In how many different ways can the letters of the word MISSISSIPPI be arranged if the four Ss cannot appear consecutively?
7. How many sequences $(a_1, a_2, \dots, a_{12})$ are there consisting of four 0s and eight 1s, if no two consecutive terms are both 0s?
8. A box is filled with three blue socks, three red socks, and four chartreuse socks. Eight socks are pulled out. In how many different sets of socks can this result? (Socks of the same color are indistinguishable.)

Exercise 2. For $u \in \mathbb{C}$ with $|u| \neq 1$, compute the integral

$$I = \int_0^{2\pi} \frac{d\theta}{1 - 2u\cos(\theta) + u^2}.$$

Hint: Rewrite the integral as a path complex integral and use that

$$cos(\theta) = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}).$$

Exercise 3. Let

$$f(z) = e^{z-2} + \frac{3\sin\big(z-2\big)}{\big(z-2\big)^2} \quad \text{and} \quad g(z) = \frac{1}{\big(z-2\big)^2}.$$

1. Find the type of singularity of f and g around $z = 2$.
2. Compute $\mathrm{Res}\big(\frac{f(z)}{z^{n+1}}, 2\big)$ and $\mathrm{Res}\big(\frac{g(z)}{z^{n+1}}, 2\big)$, for all $n \in \mathbb{N}$.

Exercise 4. In this exercise we prove the following nice characterization of power series with non-negative real coefficients (extremely used in combinatorics).

Theorem 1 (Pringsheim's Theorem). *Let $f(z) = \sum f_n z^n$ a power series of radius of convergence $R \in (0, \infty)$ with* **non-negative real coefficients**. *Then f has a singularity in $z_0 = R$.*

1. By contradiction suppose that f is analytic at R. Therefore it is analytic in a disk $D(R, r)$ centered in R of radius r, for some $r > 0$. Compute the series expansion of f around $z_0 = R - h$ for $0 < h < r/3$ using the "derivative formula for coefficients" applied to $f(z) = \sum f_n z^n$. (Why can we do it?)
2. Substitute $z = R + h$ in the expression obtained in the previous step (Why can we do it?). Find a contradiction.