Solutions 2

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Exercise 1

1. $\mathbb{E}[\delta_{I_1}(\sigma^n)\cdots\delta_{I_k}(\sigma^n)] = ?$, with $I_1,\dots,I_k$ distinct 2-element sets of $[n]$.

Note that, if $I_s = \{i_s, j_s\}$ $\forall s \in \{1,\dots,k\}$, then

$$\delta_{I_1}(\sigma^n)\cdots\delta_{I_k}(\sigma^n) = \begin{cases} 1 & \text{if } \sigma(i_s)=j_s \ \& \ \sigma(j_s)=i_s \ \forall s\in\{1,\dots,k\} \\ 0 & \text{otherwise} \end{cases}$$

Note also that if there exist $s,s'\in\{1,\dots,k\}$ s.t. $I_s \cap I_{s'} \neq \emptyset$, i.e. w.l.o.g. $I_s=\{i,j\}$ and $I_{s'}=\{i,k\}$ with $j\neq k$ (since I_s and $I_{s'}$ are distinct by assumption), then the conditions

conflict

$$\sigma(i)=j,\ \sigma(j)=i \quad \& \quad \sigma(i)=k,\ \sigma(k)=i$$

can NOT be verified, and so if $I_1,\dots,I_k$ are not disjoint then

$$\delta_{I_1}(\sigma^n)\cdots\delta_{I_k}(\sigma^n)=0.$$

$$\Rightarrow \mathbb{E}[\delta_{I_1}(\sigma^n)\cdots\delta_{I_k}(\sigma^n)] = \mathbb{P}(\sigma(i_s)=j_s,\ \sigma(j_s)=i_s,\ \forall s\in\{1,\dots,k\}) = \begin{cases} \frac{(n-2k)!}{n!} & \text{if } I_1,\dots,I_k \text{ are disjoint} \\ 0 & \text{otherwise} \end{cases}$$

2. We want to count the number of sets $\{I_1,\dots,I_k\}$ of distinct 2-elem. sets. Obviously $2k\le n$. We can create such a set in the following way: order the numbers from 1 to n in a line.

Example: 1 2 3 4 ⋯⋯ n

and then take two identical sets of k balls (numbered from 1 to k) and distribute them to among the numbers in the line.

Example:
$$\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & n-2 & n-1 & n \\ 1 & 1 & & 2 & 3 & 2 & 4 & & 3 & 4 & \end{array}$$

then the corresponding set is the one containing the k 2-elem. sets formed by the pairs of numbers having the same balls.

Example: $\{\{1,2\},\{4,6\},\{5,n-2\},\{7,n-1\}\}$

Remark: There are exactly $k!$ different balls ordering that gives the same set.

Example:
$$\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & n-2 & n-1 & n \\ 1 & 1 & & 2 & 3 & 2 & 4 & & 3 & 4 & \end{array}$$

$$\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & n-2 & n-1 & n \\ 2 & 2 & & 1 & 3 & 1 & 4 & & 3 & 4 & \end{array}$$

Give the same set!

Therefore we can conclude that the number of sets $\{I_1, \dots, I_k\}$ of distinct 2-elem. sets is

$$\underbrace{\binom{n}{2k}}_{\text{choices for the positions of the balls}} \cdot \underbrace{\frac{(2k)!}{2^k}}_{\text{\# permutations of the balls}} \cdot \underbrace{\frac{1}{k!}}_{\text{\# ordering that give the same set}} = \frac{n!}{\cancel{(2k)!}(n-2k)!} \, \frac{\cancel{(2k)!}}{2^k\, k!} = \frac{n!}{k!\,(n-2k)!\,2^k}.$$

3. We consider the function
$$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \sum_{(I_1,\dots,I_k)} \delta_{I_1}(\sigma) \cdots \delta_{I_k}(\sigma)$$

- If σ has no 2-cycles the function is trivially 0.
- If σ has $m \geq 1$ 2-cycle:

$$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \sum_{(I_1,\dots,I_k)} \delta_{I_1} \cdots \delta_{I_k}(\sigma) = \sum_{k=1}^{m} (-1)^k \binom{m}{k} = (1-1)^m - 1 = -1$$

Using that we have:

$$\mathbb{P}(C_2^n = 0) = 1 - \mathbb{P}(\sigma^n \text{ has a 2-cycle}) = 1 - \mathbb{E}\left[\mathbb{1}_{\{\sigma^n \text{ has a 2-cycle}\}}\right]$$

$$= 1 + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \sum_{\{I_1,\dots,I_k\}} \underbrace{\mathbb{E}\left[\delta_{I_1}(\sigma^n)\cdots\delta_{I_k}(\sigma^n)\right]}_{\frac{(n-2k)!}{n!} \ \text{(by 1)}} =$$

by 2

$$= 1 + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \frac{\cancel{n!}}{k!\,\cancel{(n-2k)!}\,2^k} \frac{\cancel{(n-2k)!}}{\cancel{n!}} = 1 + \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!\,2^k}$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1/2)^k}{k!} =$$

4. $$\mathbb{P}(C_2^n = r) = \sum_{\{I_1,\dots,I_r\}} \overbrace{\mathbb{P}\left(\sigma^n(i_s) = j_s \ \forall s \in \{1,\dots,r\},\ \sigma^n \text{ has no other cycles}\right)}^{\overbrace{\mathbb{P}(\sigma^n(i_s)=j_s \ \forall s\in\{1,\dots,r\})}^{=\frac{(n-2r)!}{n!}}\ \mathbb{P}(C_2^{n-2r}=0)}$$

$$= \underbrace{\frac{\cancel{n!}}{r!\,\cancel{(n-2r)!}\,2^r}}_{\substack{\#\text{ of indices}\\ \text{in the sum}\\ \text{(see point 2)}}} \cdot \frac{\cancel{(n-2r)!}}{\cancel{n!}}\ \mathbb{P}(C_2^{n-2r} = 0) = \frac{1}{r!\,2^r} \left(\sum_{k=0}^{\lfloor \frac{n-2r}{2} \rfloor} \frac{(-\frac{1}{2})^k}{k!}\right)$$

We conclude $C_2^n \xrightarrow[n\to\infty]{} C_2 \sim \text{Poisson}(1/2)$ by noting that:

$$\varphi_{C_2^n}(t) = P_{C_2^n}(e^{it}) = \sum_{r=0}^{\infty} \mathbb{P}(X_n = r)\, e^{irt} = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{1}{r!\,2^r} \left(\sum_{k=0}^{\lfloor \frac{n-2r}{2} \rfloor} \frac{(-\frac{1}{2})^k}{k!}\right) e^{irt}$$

$$\xrightarrow[n\to\infty]{} \sum_{r=0}^{\infty} \frac{1}{r!\,2^r} \left(\sum_{k=0}^{\infty} \frac{(-\frac{1}{2})^k}{k!}\right) e^{irt} = e^{-1/2} \exp(e^{it}/2) = \exp\left(\tfrac{1}{2}(e^{it} - 1)\right) = \varphi_{C_2}(t).$$

5. Fix $\ell \in \mathbb{N}$. Given a sequence of ℓ distinct numbers $I = \{i_1, \dots, i_\ell\}$ with $i_j \in [n]$, we define

$$\delta_I(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ has a } \ell\text{-cycle among } \{i_1,\dots,i_\ell\}. \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_I(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ has a } \ell\text{-cycle among } \{i_1, \dots, i_\ell\}. \\ 0 & \text{otherwise} \end{cases}$$

- $\mathbb{E}\left[\delta_{I_1}(\sigma) \cdots \delta_{I_k}(\sigma)\right] = \dfrac{(n-k\ell)!}{n!}\left((\ell-1)!\right)^k \mathbb{1}_{\{I_1,\dots,I_\ell \text{ disjoint}\}}$

- There are $\underbrace{\binom{n}{k\ell}}_{\text{choices for the positions of the balls}} \cdot \underbrace{\dfrac{(k\ell)!}{(\ell!)^k}}_{\text{\# permutations of the balls}} \cdot \underbrace{\dfrac{1}{k!}}_{\text{\# ordering that give the same set}} = \dfrac{n!}{(n-k\ell)!\,(\ell!)^k\,k!}$

- $\mathbb{P}(C_\ell^n = 0) = 1 - \mathbb{P}(\sigma^n \text{ has an } \ell\text{-cycle}) = 1 - \mathbb{E}\left[\mathbb{1}_{\{\sigma^n \text{ has an } \ell\text{-cycle}\}}\right]$

$$= 1 + \sum_{k=1}^{\lfloor n/\ell \rfloor} (-1)^k \sum_{\{I_1,\dots,I_k\}} \underbrace{\mathbb{E}\left[\delta_{I_1}(\sigma^n) \cdots \delta_{I_k}(\sigma^n)\right]}_{\frac{(n-\ell k)!}{n!}((\ell-1)!)^k} =$$

$$= 1 + \sum_{k=1}^{\lfloor n/\ell \rfloor} (-1)^k \frac{\cancel{n!}}{\cancel{(n-k\ell)!}\,(\ell!)^k\,k!} \frac{\cancel{(n-k\ell)!}}{\cancel{n!}} ((\ell-1)!)^k = 1 + (\ell-1)! \sum_{k=1} \frac{(-1)^k}{k!\,(\ell)^k}$$

by 2

$$= \sum_{k=0}^{\lfloor n/\ell \rfloor} \frac{(-1/\ell)^k}{k!}$$

$I_s = (i_1^s, \dots, i_\ell^s)$

- $\mathbb{P}(C_\ell^n = r) = \sum_{\{I_1,\dots,I_r\}} \overbrace{\mathbb{P}\left(\sigma^n(i_j^s) = i_{j+1}^s \ \forall s \in \{1,\dots,r\},\ \forall j \in [\ell],\ \sigma^n \text{ has no other cycles}\right)}^{= \frac{(n-\ell r)!}{n!}((\ell-1)!)^r\, \mathbb{P}(C_\ell^{n-\ell r} = 0)}$

$$= \underbrace{\frac{\cancel{n!}}{\cancel{(n-\ell r)!}\ \ell^r\, r!}}_{\text{\# of indices in the sum}} \frac{\cancel{(n-\ell r)!}}{\cancel{n!}}\, \mathbb{P}(C_\ell^{n-\ell r} = 0) = \frac{1}{r!\,\ell^r} \left(\sum_{k=0}^{\lfloor \frac{n-\ell r}{\ell} \rfloor} \frac{(-1/\ell)^k}{k!} \right)$$

We conclude, with the same computations as before, that

$$C_\ell^n \xrightarrow{d} C_\ell \sim \text{Poisson}(1/\ell).$$

Exercise 2

1. Let X_n be the number of fixed points in a uniform perm. σ^n of size n.

$$P_{X_n}(u) = \sum_{k=0}^{\infty} \underbrace{\mathbb{P}(X_n = k)}\, u^k = \circledast$$

$$= \mathbb{P}(X_n = k,\ X_{n-1} = k+1) + \mathbb{P}(X_n = k,\ X_{n-1} = k-1) + \mathbb{P}(X_n = k,\ X_{n-1} = k)$$

since adding a new element to σ^{n-1} we can change the value X_n by $+1, -1, 0$.

Note that

$$\mathbb{P}(X_n = k-1 \mid X_{n-1} = k) = \frac{k}{n}$$ (I have to add the new element in a cycle of length 1)

$$\mathbb{P}(X_n = k+1 \mid X_{n-1} = k) = \frac{1}{n}$$ (I have to create a new cycle)

$$\mathbb{P}(X_n = k \mid X_{n-1} = k) = 1 - \frac{1}{n} - \frac{k}{n} = \frac{n-k-1}{n}$$

$$\circledast = \underbrace{\sum_{k=0}^{\infty} \mathbb{P}(X_{n-1} = k+1)\frac{k+1}{n} u^k}_{= \frac{1}{n}\sum_{k'=1}^{\infty} \mathbb{P}(X_{n-1} = k')\, k' u^{k'-1} \;=\; \frac{1}{n} P'_{X_{n-1}}(u)} + \underbrace{\sum_{k=0}^{\infty} \mathbb{P}(X_{n-1} = k-1)\frac{1}{n} u^k}_{= \frac{u}{n} P_{X_{n-1}}(u)} + \underbrace{\sum_{k=0}^{\infty} \mathbb{P}(X_{n-1} = k)\frac{n-k-1}{n} u^k}_{= P_{X_{n-1}}(u) - \frac{u}{n} P'_{X_{n-1}}(u) - \frac{1}{n} P_{X_{n-1}}(u)}$$

$k' = k+1$

$$= P_{X_{n-1}}(u) + \frac{u-1}{n}\left(P_{X_{n-1}}(u) - P'_{X_{n-1}}(u) \right).$$

2. We want to show by induction that

$$P_{X_n}(u) = \sum_{k=0}^{n} \frac{(u-1)^k}{k!}.$$

$\boxed{n=1}$ $\quad P_{X_1}(u) = 1 + u - 1 = u \quad$ OK since trivially $\mathbb{P}(X_1 = 1) = 1$, $\mathbb{P}(X_1 = 0) = 0$

$\boxed{\text{induction}}$ Suppose that $P_{X_{n-1}}(u) = \sum_{k=0}^{n-1} \frac{(u-1)^k}{k!}$. Then by 1

$$P_{X_n}(u) = \sum_{k=0}^{n-1} \frac{(u-1)^k}{\ldots} \quad \left(\sum_{k=0}^{n-1} \frac{(u-1)^k}{\ldots} \quad \sum_{k=1}^{n-1} \frac{(u-1)^{k-1}}{\ldots} \right)$$

$$P_{X_n}(u) = \sum_{k=0}^{n-1} \frac{(u-1)^k}{k!} + \frac{(u-1)}{n}\left(\sum_{k=0}^{n-1} \frac{(u-1)^k}{k!} + \sum_{k=0}^{n-1} \frac{(u-1)^{k-1}}{(k-1)!} \right)$$

$$= \sum_{k=0}^{n-1} \frac{(u-1)^k}{k!} + \frac{1}{n} \underbrace{\sum_{k=0}^{n-1} \frac{(u-1)^{k+1}}{k!}}_{\sum_{k=1}^{n} \frac{(u-1)^k}{(k-1)!}} + \frac{1}{n} \sum_{k=0}^{n-1} \frac{(u-1)^k}{(k-1)!} = \sum_{k=0}^{n} \frac{(u-1)^k}{k!} \quad \square$$

$$= \frac{1}{n} \frac{(u-1)^n}{(n-1)!} = \frac{(u-1)^n}{n!}$$

Exercise 3

See the video "A colorful unsolved problem" posted on my web-page.