Solutions 3

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Exercise 1

1) Recall that the number of subsets of a set with n elements is 2^n (Why?)
The number of subsets of $[10]$ that contain at least one odd integer is the number of subsets of $[10]$ minus the number of subsets of $[10]$ that do not contain any integer, that is

$$2^{10} - 2^5 = 2^5(2^5-1) = 32 \cdot 31 = 992$$

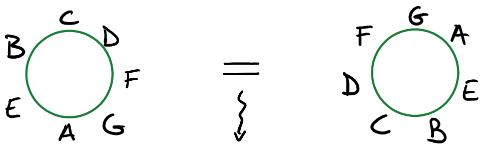
2^{10}: # subsets of $[10]$ $\quad$ 2^5: # subset of $\{2,4,6,8,10\}$

2) Note that we have $7!$ possible permutations of the 7 people (called A, B, C, D, E, F, G).

Two permutations give the same arrangement around a circle if one can be obtained from the other

- moving the first k elements at the end of the permutations

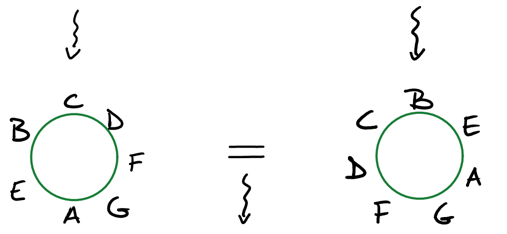
example: $\underbrace{\text{CDF}}_{k=3}\text{GAEB}$ $\quad$ GAEB$\underline{\text{CDF}}$



each person has the same neighbors

- Reversing the order of the permutation

example: CDFGAEB $\quad$ BEAGFDC



each person has the same neighbors (not necessarily on the same side)

Therefore we have

perm. of 7 elem.

$$\frac{7!}{7\cdot 2} = \frac{6!}{2} = 6\cdot 5\cdot 4\cdot 3 = 360$$

# of possible choices for K

# of possible "reverse"

possible arrangements of 7 people around a circle.

3) $$\#\{\sigma:[6]\to[6] \text{ s.t. } \sigma(1)\neq 2\} = \#\{\sigma:[6]\to[6]\} - \#\{\sigma:[6]\to[6] \text{ s.t. } \sigma(1)=2\}$$
$$= 6! - 5! = 600$$

4) We have four men (M_1, M_2, M_3, M_4) and six women $(W_1, W_2, W_3, W_4, W_5, W_6)$

- We first chose which are the four women that are married. We can do that in

$$\binom{6}{4} = \frac{6!}{4!\,2!} = \frac{6\cdot 5}{2} = 15$$

different ways.

- Then we chose which man marries which woman. We can do that is $4! = 4\cdot 3\cdot 2 = 24$ ways.

Therefore we have $\binom{6}{4}\cdot 4! = 15\cdot 24 = 360$ possible ways.

5) We first suppose that the 5 groups are distinguished (say team

A, B, C, D, E). Then "divide the 10 people in the 5 teams of 2" is equivalent to "permute the letters AABBCCDDEE". Indeed, given a permutation, we associate to the team A the people i, j, where i, j are the indexes of the letters A in the permutation.

example:

$$\overset{1}{A}\overset{2}{C}\overset{3}{A}\overset{4}{B}\overset{5}{B}\overset{6}{C}\ \overset{7}{E}\ \overset{8}{D}\overset{9}{E}\overset{10}{D}$$

$$\Downarrow$$

team A: people 1,3
team B: people 4,5
⋮

Therefore we can split 10 people in 5 distinguished teams of 2 in $\frac{10!}{(2)^5}$ ways. Moreover, we can split 10 people in 5 undistinguished teams of 2 in $\frac{10!}{2^5 \cdot 5!} = 945$ ways

↳ # of perm. of the names of the teams

6) $\#\{\text{arrangements of MISSISSIPPI s.t. no 4 consecutive S}\}$

$$= \underbrace{\#\{\text{arrangements of MISSISSIPPI}\}}_{= \frac{11!}{4!\,2!\,4!}} - \underbrace{\#\{\text{arrangements of MISSISSIPPI s.t. 4 consecutive S}\}}_{= \frac{8!}{2!\,4!}}$$

$= \frac{11!}{4!\,2!\,4!}$ → # letters; 4! → # of S, 2! → # of P, 4! → # of I

We suppose that SSSS form a unique letter

$= \frac{8!}{2!\,4!}$ → # of letters; 2! → # of S, 4! → # of P

$$= \frac{11 \cdot 10 \cdot 9 \cdot \cancel{8} \cdot 7 \cdot \cancel{6} \cdot 5}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{2}} - \frac{8 \cdot 7 \cdot \overset{3}{\cancel{6}} \cdot 5}{\cancel{2}} = 33810$$

7) In order to compute the number of sequences $(a_1, \dots, a_{12})$ such that there are four 0's and eight 1's with no consecutive 0's, we can just count in how many ways we can chose 4 positions in a sequence of eight 1's $(\overset{\text{Position 1}}{\vee} 1 \underset{\text{Position 2}}{\wedge} 1 \overset{\text{Position 3}}{\vee} 1 \wedge 1 \vee 1 \wedge 1 \vee 1 \wedge 1 \overset{\text{Position 9}}{\vee})$ where to put the

in a sequence of eight 1's (Position 1, Position 2, Position 3, ..., Position 9: $\vee 1 \wedge 1 \vee 1 \wedge 1 \vee 1 \wedge 1 \vee 1 \wedge 1 \vee$) where to put the zeros. We can obviously do that in $\binom{9}{4} = 126$ ways.

8) <u>10 socks</u>: B B B R R R C C C C

8 socks are pulled out, therefore at the end just 2 socks are in the box. The possible socks in the box are:

(a) BB (d) BR

(b) RR (e) RC

(c) CC (f) BC

Knowing which socks are in the box at the end we can compute in how many ways we could pull out the other 8 socks:

(a) We pulled out: BRRRCCCC. There are $\frac{8!}{3!\,4!}$ ways to do that

(b) We pulled out: BBBRCCCC. There are $\frac{8!}{3!\,4!}$ ways to do that

(c) We pulled out: BBBRRRCC. There are $\frac{8!}{3!\,3!\,2!}$ ways to do that

(d) We pulled out: BBRRCCCC. There are $\frac{8!}{2!\,2!\,4!}$ ways to do that

(e) We pulled out: BBBRRCCC. There are $\frac{8!}{3!\,3!\,2!}$ ways to do that

(f) We pulled out: BBRRRCCC. There are $\frac{8!}{2!\,3!\,3!}$ ways to do that

Therefore there are

$$2 \cdot \frac{8!}{3!\,4!} + 3\,\frac{8!}{3!\,3!\,2!} + \frac{8!}{2!\,2!\,4!} = 2660$$

possible ways to pull out the 8 socks.

<u>How to compute residue:</u>

$f : U \setminus \{z_0\} \longrightarrow \mathbb{C}$ holom., $z_0 \in U$.

1) If f has a removable sing. in z_0 $\Rightarrow Res(f, z_0) = 0$

2) If f has a simple pole in $z_0 \Rightarrow Res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$

3) If f has a pole of order m in $z_0 \Rightarrow Res(f, z_0) = \dfrac{g^{(m-1)}(z_0)}{(m-1)!}$

where $g(z) = (z-z_0)^m f(z)$

Proof: 1) Is trivial

2) Since z_0 is a simple pole $\Rightarrow f(z) = \sum_{n \geq -1} c_n (z-z_0)^n$ around z_0. Multipling by $(z-z_0)$ we have

$$(z-z_0) f(z) = \sum_{n \geq -1} c_n (z-z_0)^{n+1} = \sum_{p \geq 0} c_{p-1} (z-z_0)^p$$

and so $\lim_{z \to z_0} (z-z_0) f(z) = c_{-1} = Res(f, z_0)$.

3) Since z_0 is a pole of order $m \Rightarrow f(z) = \sum_{n \geq -m} c_n (z-z_0)^n$ around z_0. Multiplying by $(z-z_0)^m$ gives

$$g(z) = (z-z_0)^m f(z) = \sum_{n \geq -m} c_n (z-z_0)^{n+m} = \sum_{p \geq 0} c_{p-m} (z-z_0)^p$$

$$\Rightarrow c_{-1} = \left[(z-z_0)^{m-1}\right] g(z) = \frac{g^{(m-1)}(z_0)}{(m-1)!}$$

$\square$

Exercise 2

$u \in \mathbb{C}$ $|u| \neq 1$. We want to compute:

$$I = \int_0^{2\pi} \frac{d\theta}{1 - 2u\cos(\theta) + u^2}$$

Recall that
$$\boxed{\int_{\partial D(0,1)} \frac{f(z)}{iz} \, dz = \int_0^{2\pi} f(e^{i\theta}) \, d\theta}$$

Set
$$f(z) := \frac{1}{1 - 2u \cdot \frac{1}{2}\left(z + \frac{1}{z}\right) + u^2} = \frac{1}{1 - uz - \frac{u}{z} + u^2} = \frac{z}{z - uz^2 - u + u^2 z} = \frac{z}{-uz^2 + (u^2+1)z - u}$$

$$f(e^{i\theta}) = \frac{1}{1-2\mu\underbrace{\tfrac{1}{2}(e^{i\theta}+e^{-i\theta})}_{\cos(\theta)}+\mu^2} = \frac{1}{1-2\mu\cos\theta+\mu^2}$$

Therefore $I = \int_{\partial D(0,1)} \frac{f(z)}{iz}\,dz = \frac{1}{i}\int_{\partial D(0,1)} \overbrace{\frac{1}{-\mu z^2+(\mu^2+1)z-\mu}}^{g(z)}\,dz$

We find the singularities of $g(z)$:

$$-\mu z^2+(\mu^2+1)z-\mu=0$$

$$z_{1/2} = \frac{-\mu^2-1\pm\sqrt{\mu^4+1+2\mu^2-4\mu^2}}{-2\mu} = \frac{-\mu^2-1\pm(\mu^2-1)}{-2\mu} = \begin{cases}\mu\\ \mu^{-1}\end{cases}$$

Therefore,

$$I \underset{\substack{\text{(Residue}\\ \text{theorem)}}}{=} \frac{1}{i}2\pi i\left(\mathrm{Res}(g,\mu)\underbrace{n_{\partial D(0,1)}(\mu)}_{=\mathbb{1}_{\{|\mu|<1\}}}+\mathrm{Res}(g,\mu^{-1})\underbrace{n_{\partial D(0,1)}(\mu^{-1})}_{=\mathbb{1}_{\{|\mu|>1\}}}\right).$$

Since μ and μ^{-1} are simple poles of $g(z)=\frac{1}{-\mu(z-\mu)(z-1/\mu)}$ we have

$$\mathrm{Res}(g,\mu) = \lim_{z\to\mu}(z-\mu)g(z) = \frac{1}{-\mu\left(\mu-\frac{1}{\mu}\right)} = \frac{1}{-\mu^2+1},$$

$$\mathrm{Res}(g,\mu^{-1}) = \lim_{z\to\mu^{-1}}(z-\mu^{-1})g(z) = \frac{\,}{-\mu(\mu^{-1}-\mu)} = \frac{1}{\mu^2-1}.$$

We conclude that

$$I = 2\pi\left(\frac{1}{1-\mu^2}\mathbb{1}_{\{|\mu|<1\}}+\frac{1}{\mu^2-1}\mathbb{1}_{\{|\mu|>1\}}\right)$$

Exercise 3:

- $f(z) = e^{z-2}+\frac{3\sin(z-2)}{(z-2)^2}$

$$\lim_{z\to 2}\left(e^{z-2}+\frac{3\sin(z-2)}{(z-2)^2}\right) = \lim_{z\to 2}\ \underbrace{e^{z-2}}_{\downarrow\ 1}+3\underbrace{\frac{\sin(z-2)}{(z-2)}}_{\downarrow}\ \underbrace{\frac{1}{(z-2)}}_{\downarrow\ \pm\infty} = +\infty \Rightarrow 2 \text{ is a pole}$$

$$\lim_{z\to 2}\left(e^{z-2} + 3\,\frac{\sin(z-2)}{(z-2)^2}\right) = \lim_{z\to 2}\ \underset{\substack{\downarrow\\ 1}}{e^{z-2}} + 3\,\underset{\substack{\downarrow\\ 1}}{\frac{\sin(z-2)}{(z-2)}}\ \underset{\substack{\downarrow\\ +\infty}}{\frac{1}{(z-2)}} = +\infty \Rightarrow 2 \text{ is a pole}$$

$$\lim_{z\to 2}\ (z-2)\,f(z) = \lim_{z\to 2}\ \underset{\substack{\downarrow\\ 0}}{e^{z-2}(z-2)} + 3\ \underset{\substack{\downarrow\\ 1}}{\frac{\sin(z-2)}{(z-2)}} = 3 \Rightarrow 2 \text{ is a simple pole for } f$$

Note that $z_0 = 2$ is also a simple pole for $f(z)/z^{n+1}$ and so

$$\operatorname{Res}\left(f(z)/z^{n+1},\ 2\right) = \lim_{z\to 2}\ (z-2)\,f(z)/z^{n+1} = \frac{3}{2^{n+1}}$$

- $g(z) = \dfrac{1}{(z-2)^2}$

  Since $\lim_{z\to 2} (z-2)^2 g(z) = 1$ then $z_0 = 2$ is a double pole of $g(z)$ and also of $g(z)/z^{n+1}$ and so

$$\operatorname{Res}\left(g(z)/z^{n+1},\ 2\right) = \lim_{z\to 2}\left((z-2)^2\frac{g(z)}{z^{n+1}}\right)' = \lim_{z\to 2}\left(1/z^{n+1}\right)' = \lim_{z\to 2}\ -(n+1)\frac{1}{z^{n+2}}$$

$$= -(n+1)\,\frac{1}{2^{n+2}}.$$

Exercise 4: Proof Pringsheim's theorem

Suppose by contradiction that $f(z)$ is analytic at R, i.e., there exists a disc of radius $r>0$ centered at R where f is analytic. We choose h such that $0<h<\frac{r}{3}$ and we consider the expansion of $f(z)$ around $z_0 = R-h$

$$(*)\qquad f(z) = \sum_{m\geq 0} g_m (z-z_0)^m$$

expansion around 0

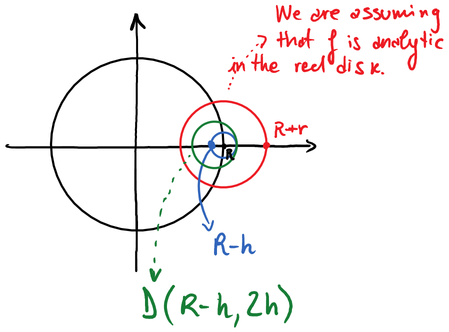
Since the coefficient of the series expansion around z_0 of an analytic function $f(z) = \sum_{n\geq 0} f_n z^n$ are given by the formula $g_m = \dfrac{f^{(m)}(z_0)}{m!}$ we have that

$$g_m = \frac{\sum_{n\geq m} f_n\ n\cdot(n-1)\cdots(n-m)\,z_0^{n-m}}{m!} = \sum_{n\geq 0} f_n \binom{n}{m} z_0^{n-m}$$

↳ $\binom{n}{m} = 0$ if $m > n$

and so $g_m \geq 0$ (since $f_n \geq 0$). Thanks to our choice for h, the series

(*) converges at $z = R+h$, therefore



$$f(R+h) = \sum_{m\geq 0}\sum_{n\geq 0} \underbrace{\binom{n}{m} f_n (R-h)^{n-m} (2h)^m}_{\geq 0}$$

$$= \sum_{n\geq 0} f_n \left(\sum_{m\geq 0} \binom{n}{m} (R-h)^{n-m} (2h)^m \right)$$

Binomial formula

$$= \sum_{n\geq 0} f_n \left[(R-h) + (2h) \right]^n = \sum_{n\geq 0} f_n (R+h)^n$$

This is a contradiction with the fact that the radius of convergence of f is R.