Random combinatorial structures
Exercise sheet nb. 4

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Exercise 1. Prove the following lemma that we already saw during the lecture.

Lemma. Let $\mathcal{C}$ be combinatorial class without element of size 0 *and set $\mathcal{A} = Seq(\mathcal{C})$. Then*

$$A(z,u) = \frac{1}{1 - C(z,u)}.$$

Why $1 - C(z,u)$ is invertible as power series?

Exercise 2. The goal of this exercise is to prove a central limit theorem for the number of parts in compositions. We recall that a *composition* of an integer n in *kparts* is a way of writing n as the sum of a sequence of (strictly) positive k integers. Two sequences that differ in the order of their terms define different compositions.

Example. The four compositions of 3 are:

- 1+1+1 (3 parts)
- 2+1 (2 parts)
- 1+2 (2 parts)
- 3 (1 part)

1. Find a way to represent every composition of n in k part as a permutation of n dots and $k-1$ bars.
2. We denote with $\mathcal{C}$ the combinatorial class of compositions, with $\{\bullet\}$ the combinatorial class containing a single dot and with $\{\bullet, |\bullet\}$ the combinatorial class containing a single dot and a bar followed by a dot. Prove that the following relation holds

$$\mathcal{C} = \{\bullet\} \times \mathrm{Seq}\big(\{\bullet, |\bullet\}\big). \tag{1}$$

3. Deduce from the previous equation that the BGF for the number of parts in a composition is
$$C(z,u) = \frac{uz}{1 - z(1+u)}.$$
Hint: What is the BGF for $\{\bullet, |\bullet\}$? Note that in Equation (1) we are "undercounting " by 1 the number of parts.

4. Deduce an expression for the PGF and conclude using the quasi-power theorem.

   Hint: Recall that $\frac{1}{1-x} = \sum_{n\geq 0} x^n$.

The following exercise sums up some of the most important properties of the *Gamma function.* You can skip it and assuming all the results in order to solve Exercise 3.

Exercise 3. For z with $\Re(z) > 0$, we define
$$\Gamma(z) = \int_0^\infty x^{z-1}\, e^{-x} dx.$$
(Recall that $x^w := \exp\left[w \log(x)\right]$ is well defined for complex exponents w, when x is a positive real number.)

1. Justify that the function is well-defined,*i.e.* that the integral is convergent whenever $\Re(z) > 0$.

2. Show that Γ is holomorphic on $\{z \in \mathbb{C} : \Re(z) > 0\}$.

   Hint: use Morera's criterium (recalled below), and Fubini's theorem to exchange the integral over a triangle and the one from 0 to $+\infty$ in the definition of Γ.

   (Morera's criterion) Let $f : U \to \mathbb{C}$ continuous, U open. Assume that, for all triangles $[A; B; C; A]$ that are completely included in U,
$$\int_{[A;B;C;A]} f(z)dz = 0.$$
   Then f is holomorphic.

3. Prove that for each $z > 0$, one has $\Gamma(z+1) = z\Gamma(z)$. Conclude that $\Gamma(n) = (n-1)!$ when n is a positive integer.

4. Show that Γ admits an analytic extension to $\mathbb{C} \setminus \{0, -1, -2, \dots\}$. (Hint: use the previous question to first extend Γ to $\{z \in \mathbb{C} : \Re(z) > -1\}$, and then make an inductive proof.)

Exercise 4. In this exercise we re-derive the asymptotic normality of the number of cycles in a uniform permutation using the quasi-power theorem.

1. Starting from the formula (that we saw during the lecture) for the PGF for the number of cycles in a uniform permutation, show that

$$P_n(u) = \frac{\Gamma(u+n)}{\Gamma(u)\Gamma(n+1)}.$$

2. Conclude using the Stirling approximation $\Gamma(z+1) \sim \sqrt{2\pi z}\left(\frac{z}{e}\right)^z$ and the quasi-power theorem.