Random combinatorial structures
Exercise sheet nb. 5

Jacopo Borga

April 1st, 2019

Exercise 1. Compute the bivariate *exponential* generating functions for:

1. the number of cycles in permutations,
2. the total length of cycles in permutations.

Deduce the corresponding *probability* generating functions.

Exercise 2. The goal of this exercise is to rederive the Poisson asymptotic behavior for the number of cycles of fixed length m (see Exercise 1 in sheet 2).

1. For $m = 1$, show that the bivariate *exponential* generating function $C(z, u)$ for the number of cycles of length one (*i.e.,* fixed points) in permutations is
$$C(z, u) = \frac{\exp(z(u-1))}{1-z}.$$
2. What is the singularity of $C(z, u)$? Using the residue theorem deduce the asymptotic behavior of $[z^n]C(z, u)$.
3. Deduce the asymptotic behavior for the corresponding PGF and conclude using the continuity theorem.
4. Note that the previous result can be also obtained writing an explicit expression for the coefficients $[z^n, u^k]C(z, u)$. What are this coefficients? How can you deduce the Poisson asymptotic behavior?
5. Deduce the result for every $m \geq 1$. Start by showing that the bivariate *exponential* generating function $C_m(z, u)$ for the number of cycles of length m is
$$C_m(z, u) = \frac{\exp((u-1)\frac{z^m}{m})}{1-z}.$$
Then use similar arguments to the ones used in points 2 and 3.