Random combinatorial structures
Exercise sheet nb. 6

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Exercise 1. The goal of this exercise is to derive the asymptotic normality for the number of parts in set-compositions.

We recall that a set composition of size n is an ordered sequence of disjoint and non-empty sets (called parts) whose union is $\{1, 2, \ldots, n\}$.

Example 1. $(\{1, 3\}, \{7, 5\}, \{4, 6, 8\}, \{2\})$ is a set partition of size 8 in 4 parts.

1. Show that the bivariate *exponential* generating function $C(z, u)$ for the number of parts in set-compositions is
$$C(z, u) = \frac{1}{1 - u(\exp(z) - 1)}.$$
2. What are the singularities of $C(z, u)$? Is $C(z, u)$ meromorphic?
3. Using the residue theorem deduce the asymptotic behavior of $[z^n]C(z, u)$.
4. Deduce the asymptotic normality for the corresponding PGF and conclude using the quasi-power theorem.

Exercise 2. Define the unit n-hypercube to be the set of points $[0, 1]^n \subset \mathbb{R}^n$. For example, the unit 0-hypercube is a point, and the unit 3-hypercube is the unit cube. Define a k-face of the unit n-hypercube to be a copy of the k-hypercube on the boundary of the n-hypercube. More formally, a k-face of the unit n-hypercube is a set of the form $\prod_{1 \le i \le n} S_i$, where S_i is either$\{0\}, \{1\}$ or $[0, 1]$ for each i between 1 and n and there are exactly k indices i such that $S_i = [0, 1]$.

1. What are the number of k-faces in the unit n-hypercube? Derive the corresponding ordinary BGF.
2. Use the ordinary BGF to derive the expected value of the dimension of a random face of the unit n-hypercube.