Random combinatorial structures
Exercise sheet nb. 7

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Exercise 1 (Hoeffding inequality). Let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. real-valued random variables. Set $X_n = \sum_{i=1}^n Y_i$. Show that for each a in $\mathbb{R}$ and any positive $u > 1$, we have

$$\mathbb{P}(X_n \geq a) \leq \frac{\mathbb{E}[u^{X_1}]^n}{u^a} \quad \text{and} \quad \mathbb{P}(X_n \leq a) \leq \frac{\mathbb{E}[u^{-X_1}]^n}{u^{-a}}.$$

Exercise 2. Let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables such that

$$\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = 1/2.$$

Set $X_n = \sum_{i=1}^n Y_i$. Show that for each a in $\mathbb{R}$,

$$\mathbb{P}(X_n \geq a) = P(X_n \leq -a) \leq e^{\frac{-a^2}{2n}}. \tag{1}$$

(Hint: Note that $\frac{e^x+e^{-x}}{2} \leq e^{\frac{x^2}{2}}$.)

Deduce form (1) that if $(Z_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables such that

$$\mathbb{P}(Z_i = 1) = \mathbb{P}(Z_i = 0) = 1/2,$$

and $S_n = \sum_{i=1}^n Z_i$ then that for each a in $\mathbb{R}$,

$$\mathbb{P}(S_n - \frac{n}{2} \geq a) \leq e^{\frac{-2a^2}{n}}.$$

Exercise 3. Let T_n be a *tournament* on n vertices, that is is a directed graph (digraph) obtained by assigning a direction for each edge in an undirected complete graph on n vertices. Given an ordering $\sigma : [n] \to [n]$, we say that i, j form an *upset* if $i \to j$ but $\sigma(i) > \sigma(j)$.

You can think at the following situation: imagine a tennis tournament among n players where every player play a match against all the other players. If the player i beats j we add the directed edge $i \to j$. At the end of the tournament we obtain a complete directed graph, *i.e.* a tournament. Moreover the ATP ranking gives an ordering of the players. The event that the player i won against j but ranked below j in the ATP ranking corresponds to an upset.

1. Show that for every possible tournament on n vertices, there exist an ordering σ such that the number of upsets is at most $\frac{1}{2}\binom{n}{2}$.

2. Let now T_n be a uniform tournament on n vertices and fix an ordering $\sigma : [n] \to [n]$. Set $U_n(\sigma)$ be the number of upsets corresponding to the tournament T_n and the ordering σ. Using the previous exercise show that

$$\mathbb{P}\left(U_n(\sigma) - \frac{1}{2}\binom{n}{2} \leq -a\right) \leq e^{-\frac{4a^2}{n^2}}.$$

3. Setting $a = n^{\frac{3}{2}}\sqrt{\log(n)}$ deduce that

$$\mathbb{P}\left(\exists\sigma \text{ s.t. } U_n(\sigma) \leq \frac{1}{2}\binom{n}{2} - n^{\frac{3}{2}}\sqrt{\log(n)}\right) \leq n!n^{-4n}. \tag{2}$$

4. Conclude that there exist a tournament T on n vertices such that for every possible ordering $\sigma : [n] \to [n]$, the number of upsets in T is at least $\frac{1}{2}\binom{n}{2} - n^{\frac{3}{2}}\sqrt{\log(n)}$.

Remark 1. Note that we proved a deterministic result using a probabilistic proof.

Remark 2. The exponential bounds obtained in Exercise 2 are very useful for the union bound in Equation (2).