Random combinatorial structures
Exercise sheet nb. 8

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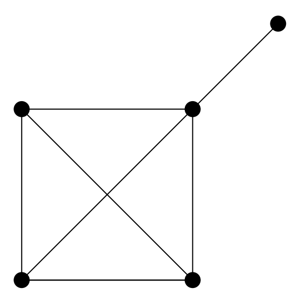
Exercise 1. Show that with probability tending to one, a uniform random permutation does not contain three consecutive entries with three consecutive increasing values (i.e. there is no i s.t. $\sigma_{i-1} + 1 = \sigma_{i+1} - 1 = \sigma_i$).

Exercise 2. Using the Stirling's formula, prove the claim from the lecture: If n and k tend simultaneously to infinity with $k = o(\sqrt{n})$, then we have

$$\binom{n}{k} \sim \frac{n^k}{k!}.$$

Exercise 3. We denote with K_4 a 4 clique. The graph H is obtained from K_4 by adding an extra vertex and edge linking this new vertex to some vertex in K_4 as shown below:



Let X_{K_4} and X_H be the number of copies of K_4 and H in the Erdős-Rényi graph $G(n, p_n)$, respectively.

1. Show that for $p_n \ll n^{-4/6}$ then $\mathbb{E}(X_{K_4})$ tends to zero. What can you conclude?
2. Show that for $p_n \gg n^{-5/7}$ then $\mathbb{E}(X_H)$ tends to infinity.
3. Find a range of values of p_n for which $\mathbb{E}(X_H)$ tends to infinity, but the probability $\mathbb{P}(X_H > 0)$ tends to 0?

Exercise 4. Prove that, for every $t, s > 0$, there exists $R(t, s)$ such that every graph with at least $R(t, s)$ vertices contains either a clique of size t or an independent set of size s. We recall that the minimal $R(t, s)$ with this property is called Ramsey number.

Hint: prove that $R(t, s) \leq R(t-1, s) + R(t, s-1)$. For that it's convenient to consider a graph G with $R(t-1, s) + R(t, s-1)$ vertices, an arbitrary vertex v in such a graph and the induced graph on $N(v)$ (neighborhood of v) and $V \setminus \{N(v) \cup \{v\}\}$. Then...