Solutions 5

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Exercise 1

BEGF for # of cycles: Let $\mathcal{P}^{cyc}$ the comb. class of cycles

We saw in class that $\mathcal{P}erm = Set(\mathcal{P}^{cyc})$.

Therefore

$$C(z,u) = Set(\underbrace{u}_{\text{every cycle increase by 1 the \# of cycles}} \mathcal{P}^{cyc}) = \exp\left(u \log \frac{1}{1-z}\right) = (1-z)^{-u} = \sum_{n=0}^{u} \binom{-u}{n}(-z)^n$$

If $X_n :=$ # of cycles in σ^n the corresponding PGF is

$$P_n(u) = \frac{[z^n]C(z,u)}{[z^n]C(z,1)} = \frac{\cancel{(-1)^n}\binom{-u}{n}}{\cancel{(-1)^n}\binom{-1}{n}} = \frac{\cancel{(-1)^{2n}}\,\frac{u(u+1)\cdots(u+n-1)}{\cancel{n!}}}{\cancel{(-1)^{2n}}\,\frac{1\cdot 2\cdots n}{\cancel{n!}}} = \frac{\Gamma(u+n)}{\Gamma(u)\,\Gamma(n+1)}$$

BEGF for length of cycles

We can describe the class of Perm as follows.

Let $\mathcal{P}_k^{cyc}$ the class of cycles of length k. Then

$$\mathcal{P}erm = Set\left(\biguplus_{k=1}^{\infty} \mathcal{P}_k^{cyc}\right) = Set\left(\underset{u}{\uparrow}\mathcal{P}_1^{cyc} \uplus \underset{u^2}{\uparrow}\mathcal{P}_2^{cyc} \uplus \underset{u^3}{\uparrow}\mathcal{P}_3^{cyc} \uplus \underset{u^4 \dots}{\uparrow}\dots\right)$$

$$\Longrightarrow C(z,u) = \exp\left(uz + u^2\frac{z^2}{2} + u^3\frac{z^3}{3} + \dots\right) = \exp\left(\log\frac{1}{1-uz}\right) = \frac{1}{1-uz} = \sum_{n\geq 0}(uz)^n$$

$X_n =$ Total cycle length of σ^n

$$P_n(u) = \frac{[z^n]C(z,u)}{[z^n]C(z,1)} = u^n$$ (obviously since every perm of length n has ...)

$$P_n(u) = \frac{[z^n]\,C(z,u)}{[z^n]\,C(z,1)} = u^n$$ (obviously since every perm of length n has total length of cycles n)

Exercise 2

1) # of cycles of length 1

$$\mathcal{P} = \mathrm{Set}\Big(u\mathcal{P}_1^{cyc} + \underbrace{\mathcal{P}_{\geq 2}^{cyc}}_{z^2/2 + z^3/3 + \dots = \log\frac{1}{1-z} - z}\Big)$$

$$\Rightarrow C(z,u) = \exp\left(zu + \log\frac{1}{1-z} - z\right) = \frac{\exp(z(u-1))}{1-z}$$

2) The only singularity is $z=1$ that is a pole of order 1, inside $\partial D(0,2)$.

Residue theorem:

$$\underbrace{\frac{1}{2\pi i}\int_{\partial D(0,2)} \frac{C(z,u)}{z^{n+1}}\,dz}_{\text{standard estimate:}} = \underbrace{\mathrm{Res}\left(\frac{C(z,u)}{z^{n+1}}, 0\right)}_{=[z^n]\,C(z,u)} + \underbrace{\mathrm{Res}\left(\frac{C(z,u)}{z^{n+1}}, e_0(u)\right)}_{=\lim_{z\to 1}(z-1)\frac{C(z,u)}{z^{n+1}} = -\exp(u-1)}$$

$$\leq \frac{1}{\cancel{2\pi i}} \cdot \cancel{4\pi}^{\,2} \cdot \sup_{z\in\partial D(0,2)} \left|\frac{C(z,u)}{z^{n+1}}\right| \leq \frac{2}{i}\,\frac{C}{2^{n+1}} = O\left(2^{-n-1}\right)$$

$$|\exp(z(u-1))| = e^{\mathrm{Re}(z(u-1))} \leq C$$

$$\Rightarrow [z^n]\,C(z,u) = \exp(u-1) + O\left(2^{-n-1}\right)$$

3) $$\Rightarrow P_n(u) = \frac{[z^n]\,C(z,u)}{[z^n]\,C(z,1)} = \exp(u-1) + O\left(\tfrac{1}{2^{n+1}}\right)$$

$$\Rightarrow \varphi_{X_n}(t) = \exp\left(e^{it} - 1\right) + o(1) \Rightarrow X_n \xrightarrow{d} \mathrm{Poisson}(1).$$

4) Set $c_{n,k} := [z^n u^k]\,C(z,u) = \frac{e^{-z}}{1-z} \cdot e^{zu}$ $\left[= \mathbb{P}(X_n = k)\right]$

4) Set $\quad c_{n,k} := [z^n u^k]\, C(z,u) = \frac{e}{1-z}\cdot e^{\cdots} \quad \left[= \mathbb{P}(X_n = k)\right]$

$$c_{n,k} = [z^n]\, \frac{z^k}{k!}\, \frac{e^{-z}}{1-z} = \left(\sum_{j=0}^{n-k} \frac{(-1)^j}{j!}\right)\frac{1}{k!} \xrightarrow[n\to\infty]{} e^{-1}/k!$$

$$\begin{cases} e^{-z} = \sum_{n\geq 0} \frac{(-1)^n}{n!} z^n \\ \frac{1}{1-z} = \sum_{n\geq 0} z^n \end{cases}$$

5) # of cycles of length m

$$C(z,u) = e^{(u-1) z^m/m} \Big/ (1-z)$$

Indeed

$$\left(z + z^2/2 + \frac{z^3}{3} + \dots\right) - \frac{z^m}{m} = \log \frac{1}{1-z} - \frac{z^m}{m}$$

$$\mathcal{P} = \mathrm{Set}\left(u\,\mathcal{P}^{cyc}_{m} + \mathcal{P}^{cyc}_{\neq m}\right)$$

$$\Longrightarrow C(z,u) = \exp\left(\frac{z^m}{m}(u-1) + \log\frac{1}{1-z}\right) = \frac{\exp\left(\frac{z^m}{m}(u-1)\right)}{1-z}$$

The only singularity is $z=1$ that is a pole of order 1, inside $\partial D(0,2)$.

<u>Residue theorem:</u>

$$\underbrace{\frac{1}{2\pi i}\int_{\partial D(0,2)} \frac{C(z,u)}{z^{n+1}}\, dz}_{\text{standard estimate:}} = \overbrace{\mathrm{Res}\left(\frac{C(z,u)}{z^{n+1}}, 0\right)}^{=[z^n]\, C(z,u)} + \overbrace{\mathrm{Res}\left(\frac{C(z,u)}{z^{n+1}}, e_0(u)\right)}^{=\lim_{z\to 1}(z-1)\frac{C(z,u)}{z^{n+1}} = -\exp\left(\frac{1}{m}(u-1)\right)}$$

$$\leq \frac{1}{\cancel{2\pi} i}\cdot \overset{2}{\cancel{4\pi}}\cdot \sup_{z\in\partial D(0,2)} \left|\frac{C(z,u)}{z^{n+1}}\right| \leq \frac{2}{i}\,\frac{C}{2^{n+1}} = O\left(2^{-n-1}\right)$$

$$\left|\exp(z(u-1))\right| = e^{\mathrm{Re}(z(u-1))} \leq C$$

$$\Longrightarrow P_n(u) = \frac{[z^n]\, C(z,u)}{[z^n]\, C(z,1)} = \exp\left(\tfrac{1}{m}(u-1)\right) + O\left(1/2^{n+1}\right)$$

$$\Rightarrow \quad \varphi_{X_n}(t) = \exp\left(\frac{1}{m}(e^{it}-1)\right) + o(1) \quad \Rightarrow \quad X_n \xrightarrow{d} \text{Poisson}(1/m).$$