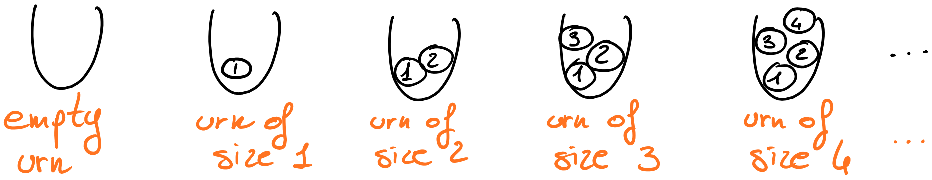
Exercise 1

1) We first consider the class of urns, i.e., the collections of balls from $\{1,\dots,n\}$.
There is exactly one urn of size n for all $n\geq 0$:



Denoting with $U(z)$ the EGF for the class of urns, we have

$$U(z) = \sum_{n\geq 0} \frac{1}{n!} z^n = e^z$$

and if $U^*(z)$ is the EGF for the class of non-empty urns we have

$$U^*(z) = e^z - 1.$$

Denoting with $\mathcal{C}$ the class of set-compositions, we trivially have that

$$\mathcal{C} = \mathrm{Seq}(\mathcal{U}^*)$$

$\hookrightarrow$ class of non-empty urns

$\left[\text{Recall that: if } \mathcal{C}=\mathcal{A}\star\mathcal{B} \text{ then } \mathcal{C}_I = \biguplus_{\substack{J,K\\ I=J\uplus K}} \mathcal{A}_J \times \mathcal{B}_K\right]$

If $C(z)$ denotes the EGF of $\mathcal{C}$, then from the previous eq. we have

$$C(z) = \frac{1}{1-(e^z-1)}.$$

Moreover, if we want to track also the numbers of parts in $\mathcal{C}$, denoting with $C(z,u)$ the corresponding BEGF, we obtain

$$C(z,u) = \frac{1}{1-u(e^z-1)}$$

2) For fixed u, the singularities of $C(z,u)$ are the $z \in \mathbb{C}$ s.t.

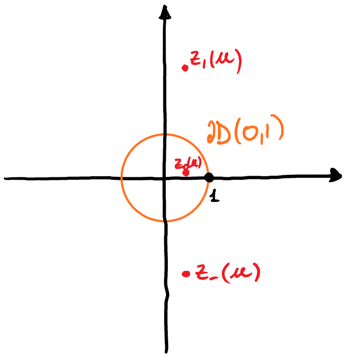
$$e^z = \frac{1}{u} + 1$$

and so

$$z_k(u) = \log\left(\frac{1}{u} + 1\right) + i 2\pi k \quad , k \in \mathbb{Z}$$

Moreover, $C(z,u)$ is meromorphic in $\mathbb{C}$ since $1 - u(e^z - 1)$ is holomorphic for every $u \in \mathbb{C}$.

3)



We fix u in a neighbourhood of 1.
Note that

$$z_k(u) \xrightarrow[u \to 1]{} \log(2) + i \cdot 2\pi k \qquad \forall k \in \mathbb{Z}$$

Therefore, tacking u in a sufficently small neighbourhood of 1, we can assume that $z_0 \in \partial D(0,2)$ and $z_k \notin \partial D(0,2)$ $\forall k \in \mathbb{Z}^*$.
Using the residue theorem, we can conclude that

$$\frac{1}{2\pi i} \int_{\partial D(0,1)} \frac{C(z,u)}{z^{n+1}} \, dz = [z^n] C(z,u) + \operatorname{Res}\left(\frac{C(z,u)}{z^{n+1}}, z_0(u)\right)$$

(1) We first determine $\operatorname{Res}\left(\frac{C(z,u)}{z^{n+1}}, \underset{\overset{\parallel}{\log\left(\frac{1}{u}+1\right)}}{z_0(u)}\right)$:

Note that $\displaystyle \frac{d}{dz}\left(z^{n+1} \cdot \left(1 - u(e^z - 1)\right)\right)\Big|_{z=z_0} = \left((n+1) z^n \left(1 - u(e^z - 1)\right) + z^{n+1}(-u e^z)\right)\Big|_{z=z_0} = z_0^{n+1} \cdot (-1-u) \neq 0$

$\Rightarrow$ z_k is a simple pole $\Rightarrow$

[Note that $C(z,u) = \dfrac{f_1(z,u)}{f_2(z,u)}$ and $(z-z_0)\,\dfrac{f_1(z)}{f_2(z)} = f_1(z)\cdot\left(\dfrac{z-z_0}{f_2(z)-\underbrace{f_2(z_0)}_{=0}}\right) \xrightarrow[z\to z_0]{} \dfrac{f_1(z)}{f_2'(z)}$]

$$\mathrm{Res}\left(\frac{C(z,u)}{z^{n+1}},\, z_0\right) = \lim_{z\to z_0} (z-z_0)\,\frac{C(z,u)}{z^{n+1}} = \frac{1}{z_0^{n+1}}\cdot\frac{1}{(-1-u)} = \frac{z_0^{-1-n}}{(-1-u)}.$$

(2) We now estimate the integral:

$$\left|\frac{1}{2\pi i}\int_{\partial D(0,1)} \frac{C(z,u)}{z^{n+1}}\,dz\right| \leq \frac{1}{2\pi}\cdot 2\pi\cdot \sup_{z\in\partial D(0,1)} \underbrace{\left|\frac{1}{z^{n+1}(1+u(e^z-1))}\right|}_{\leq C}$$

$\Longrightarrow$ the integral is $O(1)$ unif. for u in a suff. small neighb. of 1.

From (1) and (2) we obtain that

$$[z^n]\,C(z,u) = \frac{z_0^{-1-n}}{(1+u)} + O(1).$$

4). We get that the probability generating function is, uniformly for u in a suff. small neighbourhood of 1,

$$P_n(u) = \frac{(z_0(u))^{-n-1}}{(1+u)}\cdot\frac{2}{(\underbrace{z_0(1)}_{=\log 2})^{-n-1}} + O(1)$$

$$= \left(\frac{2}{1+u}\right)\left(\frac{\log(2)}{\log(1+\frac{1}{u})}\right)\cdot\left(\frac{\log(2)}{\log(1+\frac{1}{u})}\right)^n(1+o(1))$$

Setting $A(u) = \dfrac{2\log(2)}{(1+u)\log(1+\frac{1}{u})}$, $B(u) = \dfrac{\log(2)}{\log(1+\frac{1}{u})}$ and $\beta_n = n$

and noting that:

- $A(u)$ and $B(u)$ are holom. in a neighb. of 1
- $A(1) = B(1) = 1$
- $B'(u) = \dfrac{\log(2)}{z(z+1)\log^2\left(\frac{1}{z}+1\right)}$ & $B''(u) = \dfrac{\log(2)\left((2z+1)\log\left(\frac{1}{z}+1\right)-2\right)}{z^2(z+1)^2\log^3\left(\frac{1}{z}+1\right)}$

and so

$$B'(1) = \frac{1}{2\log(2)} \qquad \& \qquad B''(1) = \frac{3\log(2)-2}{4\log^2(2)}$$

$$B''(1) + B'(1) - B'(1)^2 = \frac{3\log(2)-3}{4\log^2(2)} + \frac{1}{2\log(2)}$$
$$= \frac{5\log(2)-3}{4\log^2(2)}$$

We can conclude from the quasi-power thm that if

X_n = # of parts in a unif. set-comp. of size n

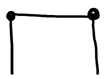
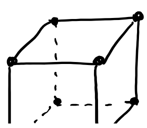
Then

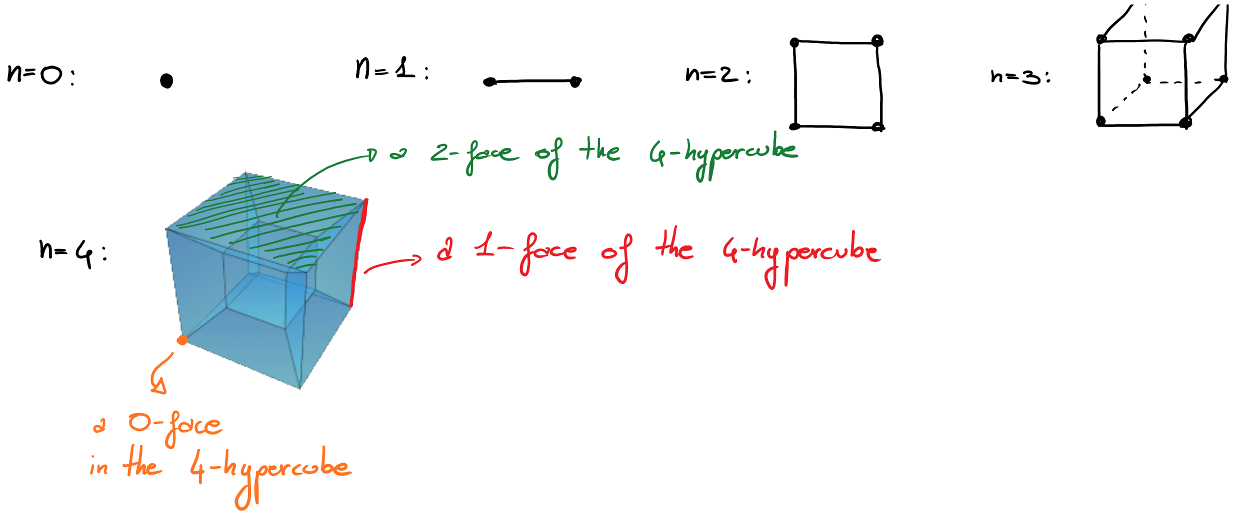
$$\mathbb{E}[X_n] = \frac{n}{2\log(2)} + O(1)$$

$$\mathrm{Var}[X_n] = \frac{5\log(2)-3}{4\log^2(2)}\, n + O(1)$$

and $$\frac{X_n - \mathbb{E}[X_n]}{\sqrt{\mathrm{Var}(X_n)}} \xrightarrow{d} \mathcal{N}(0,1)$$ $\square$

Exercise 2

$n=0$: 
$n=1$:
$n=2$: 
$n=3$: 



Let $Q(n,k)$ denotes the the number of k-faces in an n-hypercube

For every vertex in a n-hypercube there are exactly n segments (i.e., 2-hypercube) incident to this vertex. Note that any choice of k distincts segmets (incident to the same vertex) identify a k-hypercube. Therefore the number of k-faces incidend at each vertex of an n-hypercube is $\binom{n}{k}$. Moreover, noting that every k-hypercube has 2^k vertices we can conclude that

\# of verticec in a n-hypercube

\# of k-faces incident to a vertex in an n-hypercube

$$Q(n,k) = \frac{2^n \binom{n}{k}}{2^k} = 2^{n-k}\binom{n}{k}$$

↳ every k-faces has been counted 2^k times (i.e., the \# of vertices in a k-face).

The corresponding OGF is

$$Q(z,u) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} 2^{n-k}\binom{n}{k} z^n u^k = \sum_{n\geq 0}^{\infty} 2^n \underbrace{\left(\sum_{k=0}^{n}\binom{n}{k}\left(\frac{u}{2}\right)^k\right)}_{=\left(1+\frac{u}{2}\right)^n} z^n = \sum_{n=0}^{\infty} 2^n \cdot \left(1+\frac{u}{2}\right)^n z^n$$

$$= \left(1+\frac{u}{2}\right)^n$$

2) The corresponding PGF is

$$P_n(u) = \frac{[z^n]\, Q(z,u)}{[z^n]\, Q(z,1)} = \frac{\cancel{2^n}\left(1+\frac{u}{2}\right)^n}{\cancel{2^n}\left(3/2\right)^n} = \left(\frac{2}{3}\right)^n \left(1+\frac{u}{2}\right)^n$$

$$\mathbb{E}[X_n] = \left(\frac{d}{du} P_n(u)\right)\Big|_{u=1} = \left(\frac{2}{3}\right)^n n \cdot \left(\frac{3}{2}\right)^{n-1} \cdot \frac{1}{2} = \frac{n}{3} \qquad \square$$

$$P_n(u) = \quad [z^n]$$