Exercise 1

$$X_n = \sum_{i=1}^n Y_i$$

$$\mathbb{P}(X_n \geq a) \underset{\text{Chernoff bounds}}{\leq} \frac{\mathbb{E}[u^{X_n}]}{u^a} = \frac{\mathbb{E}[\prod_{i=1}^n u^{Y_i}]}{u^a} \underset{(Y_i)_i \text{ i.i.d.}}{=} \frac{\prod_{i=1}^n \mathbb{E}[u^{Y_1}]}{u^a} = \frac{\mathbb{E}[u^{Y_1}]^n}{u^a} \qquad \forall u > 1$$

The other inequality is similar.

Exercise 2

$$Y_i = \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases} \qquad X_n = \sum_{i=1}^n Y_i$$

$$\mathbb{P}(X_n \geq a) = \mathbb{P}(X_n \leq -a) \leq \frac{\mathbb{E}[u^{Y_1}]^n}{u^a} \underset{u = e^{\lambda} \text{ with } \lambda > 0}{=} \frac{\mathbb{E}[e^{\lambda Y_1}]^n}{e^{\lambda a}} = \frac{\left(\frac{e^{\lambda}+e^{-\lambda}}{2}\right)^n}{e^{\lambda a}}$$

$$\leq \frac{(e^{\lambda^2/2})^n}{e^{\lambda a}} = e^{\frac{\lambda^2}{2}n - \lambda a} = e^{\frac{n}{2}\frac{a^2}{n^2} - \frac{a^2}{n}} = e^{-a^2/2n} \qquad (1)$$

$\hookrightarrow \lambda = \frac{a}{n}$ (min of $\frac{\lambda^2 n}{2} - \lambda a$)

$$\frac{e^{\lambda}+e^{-\lambda}}{2} \leq e^{\lambda^2/2}$$

since $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$

$$\Rightarrow \frac{e^{\lambda}+e^{-\lambda}}{2} = 1 + \frac{\lambda^2}{2} + \frac{\lambda^4}{4!} + \frac{\lambda^6}{6!} + \dots + \frac{\lambda^{2k}}{(2k)!}$$

$$e^{\lambda^2/2} = 1 + \frac{\lambda^2}{2} + \frac{\lambda^4}{2!\,2^2} + \frac{\lambda^6}{3!\cdot 2^3} + \dots + \frac{\lambda^{2k}}{k!\,2^k}$$

we conclude using that $2^k k! \leq (2k)!$

Corollary:

$$Z_i = \begin{cases} 0 & p = 1/2 \\ 1 & p = 1/2 \end{cases} \qquad S_n = \sum_{i=1}^n Z_i \qquad E[S_n] = n/2$$

$$\mathbb{P}\left(S_n - \frac{n}{2} > a\right) = \mathbb{P}\left(\sum_{i=1}^n (Z_i - 1/2) > a\right) = \mathbb{P}\left(\sum_{i=1}^n \underbrace{(2Z_i - 1)}_{= \begin{cases} -1 & \text{w.p. } 1/2 \\ 1 & \text{w.p. } 1/2 \end{cases}} > 2a\right) <$$

$$< e^{-(2a)^2/2n} = e^{-2a^2/n}$$

$\Downarrow$
(1)

Exercise 3

1. Observe that by tacking σ and reverse of σ, we can have

$$\# \text{ of upsets} \leq \frac{1}{2}\binom{n}{2}$$

2. Fix $\sigma : [n] \longrightarrow [n]$, T_n is a uniform tournament, $U_n = \#$ of upset in (T_n, σ).

$$U_n = \sum_{i \neq j} Y_{ij}^n \text{ , where } Y_{ij}^n = \begin{cases} 1 & \text{if } (i,j) \text{ is an upset in } T_n \\ 0 & \text{otherwise} \end{cases}$$

It's easy to show that $\mathbb{P}(Y_{ij}^n = 0) = \mathbb{P}(Y_{ij}^n = 1) = \frac{1}{2}$. Note that $\mathbb{E}[U_n] = \frac{1}{2}\binom{n}{2}$ and so

$$\mathbb{P}\left(U_n - \frac{1}{2}\binom{n}{2} < -a\right) \leq e^{-2a^2/\binom{n}{2}} = e^{-\frac{2a^2}{\frac{n(n-1)}{2}}} < e^{-\frac{4a^2}{n^2}}$$

3. We set $a = n^{3/2}\sqrt{\log(n)}$ $\left(\begin{array}{l} \text{I want } e^{-\frac{4a^2}{n^2}} < n! \\ \Rightarrow a = n^{3/2}\sqrt{\log n} \end{array}\right)$ obtaining

$$\mathbb{P}\left(U_n < \frac{1}{2}\binom{n}{2} - n^{3/2}\sqrt{\log(n)}\right) < e^{-\frac{4}{n^2}n^3 \log(n)} = e^{-4n\log(n)} = n^{-4n}$$

and so

$$\mathbb{P}\left(\exists\, \sigma \text{ s.t. } U_n(\sigma) < \frac{1}{2}\binom{n}{2} - n^{3/2}\sqrt{\log(n)}\right) \leq \underbrace{n!}_{\substack{\#\text{ of} \\ \text{perm. of size } n.}} n^{-4n}$$

4. Using the Stirling's approximation $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ we have

$$n!\, n^{-4n} \sim \sqrt{2\pi n}\, \frac{n^n}{e^n}\, n^{-4n} = \sqrt{2\pi n}\, \frac{n^{-3n}}{e^n} \longrightarrow 0.$$

$\Rightarrow$ $n!\, n^{-4n} << 1$ $\Rightarrow$ $\exists$ a tornament T on n vertices s.t. for every possible ordering σ, the number of upsets in T is at least

$$\frac{1}{2}\binom{n}{2} - n^{3/2}\log(n).$$