Random combinatorial structures
Exercise sheet nb. 11

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May 26th, 2019

Exercise 1. In this exercise we reprove the central limit theorem for the number X_n of descents in uniform random permutation using cumulants. You can assume that (this was proved during the course)

$$\mathrm{Var}(X_n) = \frac{n}{12} + O(1).$$

1. Write X_n as a sum of indicator functions.
2. Express the r-th cumulant of X_n as a sum of simpler cumulants.
3. Note that most of the simpler cumulants are equal to 0. Which one? Why?
4. Estimate the number of simpler cumulants that are different from zero.
5. We recall the following result (that will be proved during the Friday's lecture)

   Lemma 1. *For any $r > 0$, there exists a constant $B_r > 0$ such that, for any r.v. $X_1, \dots, X_r$ bounded by* 1 *(e.g. indicators), we have*

   $$|\kappa(X_1, \dots, X_r)| \leq B_r.$$

   Using this lemma and the moment method via cumulants prove a central limit theorem for X_n.

Exercise 2. The goal of this exercise is to estimate the number of primes that divide an integer smaller than n.

In what follow p denotes a prime number. Let $x \in \mathbb{N}_{>0}$ and set

$$\nu(x) := \text{number of } p \text{ s.t. } p|x.$$

For example $\nu(12) = 2$. We want to prove the following:

Fix an arbitrary slowly increasing sequence $\omega(n) \to \infty$. Then for all but $o(n)$ integers $x \leq n$ we have that

$$|\nu(x) - \log(\log(n))| \leq \omega(n)\sqrt{\log(\log(n))}.$$

In particular

$$\nu(x) \sim \log(\log(n)).$$

We prove this result using the Second moment method.

1. Let $Z(x) := \sum_{p \leq n^{0.1}} \mathbb{1}_{\{p|x\}}$. Note that $|\nu(x) - Z(x)| \leq 10$ for all $x \leq n$.

   We now consider a uniform integer $\boldsymbol{x}$ from 1 to n.

2. Show that $\mathbb{E}[\mathbb{1}_{\{p|\boldsymbol{x}\}}] = \frac{\lfloor n/p \rfloor}{n}$.

3. Using that $\sum_{p \leq t} \frac{1}{p} = \log(\log(t)) + O(1)$, show that

   $$\mathbb{E}[Z(\boldsymbol{x})] = \log(\log(n)) + O(1).$$

   (For a proof of the estimates $\sum_{p \leq t} \frac{1}{p} = \log(\log(t)) + O(1)$ you can have a look to the PDF "EulerPrime")

4. Show that $\mathrm{Var}[Z(\boldsymbol{x})] = \mathbb{E}[Z(\boldsymbol{x})] + O(1)$.

   (Hint: first show that $\mathrm{Cov}(\mathbb{1}_{\{p|\boldsymbol{x}\}}, \mathbb{1}_{\{q|\boldsymbol{x}\}}) \leq \frac{3}{n}$ for every pair of primes $p < q$)

5. Conclude using the Second moment method.