Solutions 10

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Exercise 1

2. $X_n = \sum_{i=1}^{4n} \mathbb{1}_{\{\sigma(i) \equiv i \bmod n\}}$ where σ is a unif. perm. of size $4n$.

3. $$\mathbb{E}[(X_n)_r] = \sum_{\substack{i_1,\dots,i_r \in [4n] \\ \text{distinct}}} \mathbb{E}\Big[\underbrace{\mathbb{1}_{\{\sigma(i_1)\equiv i_1 \bmod n\}} \cdots \mathbb{1}_{\{\sigma(i_r)\equiv i_r \bmod n\}}}_{= \mathbb{P}(\sigma(i_1)\equiv i_1 \bmod n, \dots, \sigma(i_r) \equiv i_r \bmod n)}\Big]$$

4. $$\mathbb{P}(\sigma(i_1)\equiv i_1 \bmod n, \dots, \sigma(i_r)\equiv i_r \bmod n) =$$
$$= \mathbb{P}(\sigma(i_1)\equiv i_1 \bmod n) \cdot \mathbb{P}(\sigma(i_2)\equiv i_2 \bmod n \mid \sigma(i_1)\equiv i_1 \bmod n) \cdots$$
$$\circledast \qquad \cdots \mathbb{P}(\sigma(i_r)\equiv i_r \bmod n \mid \sigma(i_1)\equiv i_1 \bmod n, \dots, \sigma(i_{r-1})\equiv i_{r-1} \bmod n)$$

Now

$$\mathbb{P}(\sigma(i_1)\equiv i_1 \bmod n) = \frac{4}{4n}$$

$$\mathbb{P}(\sigma(i_2)\equiv i_2 \bmod n \mid \sigma(i_1)\equiv i_1 \bmod n) = \begin{cases} \frac{4}{4n-1} & \text{if } i_1 \not\equiv i_2 \bmod n \\ \frac{3}{4n-1} & \text{if } i_1 \equiv i_2 \bmod n \end{cases}$$

$$\Rightarrow \mathbb{P}(\sigma(i_2)\equiv i_2 \bmod n \mid \sigma(i_1)\equiv i_1 \bmod n) \le \frac{4}{4n-1}$$

Similarly, the factor

$$\mathbb{P}(\sigma(i_j)\equiv i_j \bmod n \mid \sigma(i_1)\equiv i_1 \bmod n, \dots, \sigma(i_{j-1})\equiv i_{j-1} \bmod n) \le \frac{4}{4n-j+1} \quad \left(\begin{array}{l}\text{with equality} \\ \text{when the } i_j\text{'s are} \\ \text{not congruent}\end{array}\right)$$

and so

$$\mathbb{P}(\sigma(i_1)\equiv i_1 \bmod n, \dots, \sigma(i_r)\equiv i_r \bmod n) \le \frac{4^r (4n-r)!}{(4n)!}$$

5. Note that the # of tuples $(i_1,\dots,i_r) \in [4n]^r$ s.t. some i_j's are congruent mod n is $\le \underbrace{\binom{4n}{r-1}}_{\text{choices}} \cdot \underbrace{3(r-1)}_{\text{choices for}} = O(n^{r-1})$.

$\bmod n$ is $\leq \underbrace{\binom{4n}{r-1}}_{\substack{\text{choices} \\ \text{for the first} \\ r-1 \text{ elements}}} \cdot \underbrace{3(r-1)}_{\substack{\text{choices for} \\ \text{the last element} \\ (\text{congruent to one} \\ \text{of the previous})}} = O(n^{r-1})$.

6. $$\mathbb{E}[(X_n)_r] = \sum_{\substack{i_1,\dots,i_r \in [4n] \\ \text{distinct}}} \mathbb{P}(\sigma(i_1) \equiv i_1 \bmod n, \dots, \sigma(i_r) \equiv i_r \bmod n)$$

$$= \underbrace{\sum_{\substack{i_1,\dots,i_r \in [4n] \\ \text{distinct} \\ \text{and with no} \\ \text{congruent } i_j\text{'s}}}}_{4n(4n-4)\cdots(4n-4(r-1))} \underbrace{\mathbb{P}(\sigma(i_1) \equiv i_1 \bmod n, \dots, \sigma(i_r) \equiv i_r \bmod n)}_{= \frac{4^r (4n-r)!}{(4n)!}}$$

$$+ \underbrace{\sum_{\substack{i_1,\dots,i_r \in [4n] \\ \text{distinct} \\ \text{and with} \\ \text{congruent } i_j\text{'s}}}}_{O(n^{r-1})} \underbrace{\mathbb{P}(\sigma(i_1) \equiv i_1 \bmod n, \dots, \sigma(i_r) \equiv i_r \bmod n)}_{\leq \frac{4^r (4n-r)!}{(4n)!}}$$

$$\Rightarrow \mathbb{E}\left[(X_n)_r\right] \sim 4n(4n-4)(4n-4\cdot 2)\cdots(4n-4(r-1)) \frac{4^r (4n-r)!}{(4n)!}$$

$$\Rightarrow \mathbb{E}\left[(X_n)_r\right] \xrightarrow{n\to\infty} 4^r$$

r-th factorial moment of Poisson(4)

Applying the moment method we can conclude that

$$X_n \xrightarrow{d} \text{Poisson}(4).$$

Exercise 2

We first check that $f_X(x)$ and $f_Y(x)$ are well-def prob. densities.

$$\int_0^{\infty} (1+\sin(2\pi \log(x))) \frac{1}{\sqrt{2\pi}\, x} e^{-\log(x)^2/2}\, dx =$$

$\log(x) = y$
$\begin{cases} x = e^y \\ dx = dy\, e^y \end{cases}$

$$\int_0 (1+\sin(2\pi\log(x)))\frac{1}{\sqrt{2\pi}\,x} e \quad dx = \qquad \{dx = dy\, e^y$$

$$= \int_{-\infty}^{+\infty} \left(1+\sin(2\pi y)\right)\frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}\, dy$$

$$= \underbrace{\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}\, dy}_{=1 \ \text{(Gaussian)}} + \underbrace{\int_{-\infty}^{+\infty}\sin(2\pi y)\, e^{-y^2/2}\, dy}_{=0 \ \left(\text{since } \sin(-x)=-\sin(x)\right)}$$

We now show that $\mathbb{E}[X^\rho]-\mathbb{E}[Y^\rho]=0$

$$\mathbb{E}[Y^\rho]-\mathbb{E}[X^\rho] = \int_0^{\infty} x^\rho \frac{e^{-\log(x)^2/2}}{\sqrt{2\pi}\,x}\sin(2\pi\log(x))\, dx$$

$\log(x)=y$

$$= \int_{-\infty}^{+\infty} e^{y\rho}\, \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, \sin(2\pi y)\, dy$$

$$= e^{\rho^2/2}\int_{-\infty}^{+\infty} \underbrace{\sin(2\pi y)}_{=\sin(2\pi(y-\rho))}\, e^{-(y-\rho)^2/2}\, dy = 0$$

(sin is periodic)