## Exercise 1

Let $X_n = \#\, i \in [\![2, n-1]\!]$ s.t. $\sigma_{i-1} + 1 = \sigma_{i+1} - 1 = \sigma_i$

We want to show that $\mathbb{P}(X_n = 0) \xrightarrow[n\to\infty]{} 1$.

Thanks to the $1^{st}$ moment method it is enough to show that

$$\mathbb{E}[X_n] \longrightarrow 0$$

Note that

$$X_n = \sum_{i=2}^{n-1} \mathbb{1}_{\{\sigma_{i-1}+1 = \sigma_{i+1} - 1 = \sigma_i\}}$$

$$\Rightarrow \quad \mathbb{E}[X_n] = \sum_{i=2}^{n-1} \mathbb{P}(\sigma_{i-1} + 1 = \sigma_{i+1} - 1 = \sigma_i)$$

For fixed $i \in [\![2, n-1]\!]$, we have

$$\mathbb{P}(\sigma_{i-1} + 1 = \sigma_{i+1} - 1 = \sigma_i) = \sum_{j=2}^{n-1} \underbrace{\mathbb{P}(\sigma_{i-1} + 1 = \sigma_{i+1} - 1 = \sigma_i,\ \sigma_i = j)}_{= \underbrace{\mathbb{P}(\sigma_{i-1} = j-1, \sigma_{i+1} = j+1 \mid \sigma_i = j)}_{\frac{1}{n-1}\cdot\frac{1}{n-2}} \underbrace{\mathbb{P}(\sigma_i = j)}_{\frac{1}{n}}}$$

$$= \frac{1}{n(n-1)}$$

$$\Rightarrow \quad \mathbb{E}[X_n] = \frac{n-2}{n(n-1)} \xrightarrow[n\to\infty]{} 0 \qquad \square$$

## Exercise 2

Stirling: $\quad n! \sim \frac{1}{\sqrt{2\pi n}} \cdot \left(\frac{n}{e}\right)^n$

$\sqrt{2\pi n}\quad \left(\frac{n}{e}\right)$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \sim \frac{\frac{1}{\sqrt{\cancel{2\pi} n}}\left(\frac{n}{e}\right)^n}{k!\cdot\frac{1}{\sqrt{\cancel{2\pi}(n-k)}}\cdot\left(\frac{n-k}{e}\right)^{n-k}}$$

$$= \frac{1}{k!}\sqrt{\frac{n-k}{n}}\;\frac{n^n}{(n-k)^{n-k}}\;\frac{1}{e^k}$$

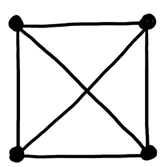
Note that if $k = o(\sqrt{n})$ then

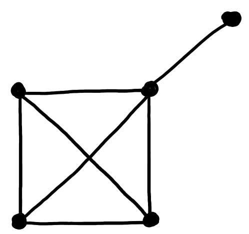
- $\sqrt{\frac{n-k}{n}} \sim 1$
- $\frac{n^n}{e^k (n-k)^{n-k}} = \frac{1}{e^k}\underbrace{\left(\frac{n}{n-k}\right)^{n-k}}_{\left(1+\frac{k}{n-k}\right)^{n-k} \sim e^k}\, n^k \sim n^k$ (since $k = o(\sqrt{n})$)

We can conclude that

$$\binom{n}{k} \sim \frac{n^k}{k!}$$

## Exercise 3

$K_4 =$ 

$H =$ 

$$X_{K_4} = \#\text{ of } K_4 \text{ in } G(n, p_n) = \sum_{\substack{J \subseteq [n] \\ |J| = 4}} \mathbb{1}_{\{G|_J = K_4\}}$$

$$X_H = \#\text{ of } H \text{ in } G(n, p_n) \geq \sum_{\substack{I \subseteq [n] \\ |I| = 5}} \mathbb{1}_{\{G|_I \supseteq H\}}$$

We first analyse the $\mathbb{E}[X_{K_4}]$ and $\mathbb{E}[X_H]$:

- $\mathbb{E}[X_{K_4}] = \sum_{\substack{J \subseteq [n] \\ |J|=4}} \underbrace{\mathbb{P}(G|_J = K_4)}_{p_n^6} = \binom{n}{4} p_n^6 \sim \frac{n^4}{4!} p_n^6$

- $\mathbb{E}[X_H] \geq \sum_{\substack{I \subseteq [n] \\ |I|=5}} \underbrace{\mathbb{P}(G|_I = H)}_{\geq p_n^7} = \binom{n}{5} p_n^7 \sim \frac{n^5}{5!} p_n^7$

1. If $p_n \ll n^{-4/6}$ then $\mathbb{E}[X_{K_4}] \ll \frac{n^4}{4!} n^{-4} = \frac{1}{4!}$

$\Rightarrow \mathbb{E}[X_{K_4}] \longrightarrow 0 \underset{\substack{1^{st}\text{ moment} \\ \text{method}}}{\Longrightarrow} \mathbb{P}(X_{K_4} = 0) \longrightarrow 1$

2. If $p_n \gg n^{-5/7}$ then $\mathbb{E}[X_H] \gg \frac{n^5}{5!} n^{-5} = \frac{1}{5!}$

$\Rightarrow$ if $n^{-5/7} \ll p_n \ll n^{-4/6}$ then we have that

- $\mathbb{E}[X_H] \longrightarrow 0$
- $\mathbb{P}(X_H > 0) \leq \mathbb{P}(X_{K_4} > 0) = 1 - \mathbb{P}(X_{K_4} = 0) \longrightarrow 0$ $\square$

## Exercise 6

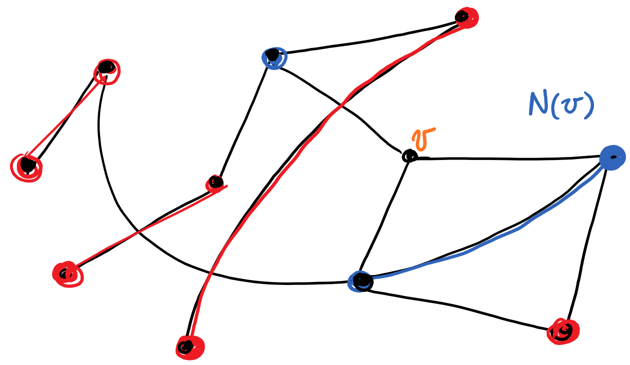
$s,t > 0$. We want to prove that $R(t,s) < \infty$ showing that

$$R(t,s) \leq R(t-1,s) + R(t,s-1)$$

<u>Proof by induction</u>:

- Obviously, for all $n \geq 1$, $R(n,1) = R(1,n) = 1 < \infty$.
- <u>Inductive step</u>:

Consider a graph $G$ with $R(t-1,s) + R(t,s-1)$ vertices and an arbitrary vertex $v \in G$. Then we considere




$G(V \setminus \{N(v) \cup \{v\}\})$

Note that

$$|V \setminus \{N(v) \cup \{v\}\}| + |N(v)| + 1 = |G| = R(t-1,s) + R(t,s-1)$$

and so either $|V \setminus \{N(v) \cup \{v\}\}| \geq R(t,s-1)$

or $|N(v)| \geq R(t-1,s)$

(indeed if $|V \setminus \{N(v) \cup \{v\}\}| < R(t,s-1)$ then

$|N(v)| = R(t-1,s) + R(t,s-1) - 1 - |V \setminus \{N(v) \cup \{v\}\}| > R(t-1,s) - 1$ )

Assume that $|N(v)| \geq R(t-1,s)$ (the other case is similar)

$\Longrightarrow$ $G(N(v))$ either contains a clicque of size $t-1$ or an independent set of size $s$.

If $G(N(v))$ contains a clicque of size $t-1$ then

$G(N(v) \cup v)$ contains a clique of size $t$.

Otherwise $G(N(v))$ contains an indep. set of size $s$ and so also the original graph $G$.

This ends the proof. $\square$