Square permutations are typically rectangular

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This talk is based on joint work with Erik Slivken [4]

We establish permuton convergence and local convergence for large uniform random square permutations. First we describe the global behavior by showing that these permutations have a permuton limit which can be characterized as a random rectangle. We also explore fluctuations about this random rectangle, which we can describe through coupled Brownian motions. Second, we consider the limiting behavior of the neighborhood of a point in the permutation through local limits. As a byproduct, we also determine the random limiting distribution of the proportion of occurrences and consecutive occurrences of any given pattern in a uniform random square permutation.

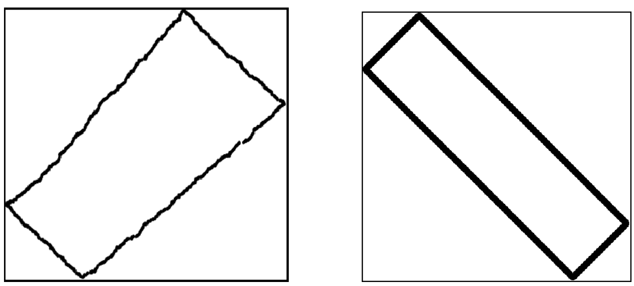

Figure 1: The diagram of two typical square permutations of size 1 000 and 1 000 000.

Square permutations

Square permutations are permutations where every point is a record, *i.e.*, a maximum or minimum, either from the left or from the right. Square permutations can be also described as a pattern-avoiding class, where the avoided patterns are all 16 patterns of length five with a point that is not a record. Mansour and Severini [7] determine the enumeration of the class proving that there are $2(n+2)4^{n-3} - 4(2n-5)\binom{2n-6}{n-3}$ square permutations of size n. This permutation class was later discussed in [5, 6, 1] (in this last paper the authors refer to square permutations as *convex permutations*). In this talk we focus on the shape of square permutations.

Sampling asymptotically uniform square permutations

The starting point for all our results is the sampling procedure described in this section. We define a projection from the set of square permutations to the set of *anchored pairs of sequences* of labels, *i.e.*, triples $(X, Y, z_0) \in \{U, D\}^n \times \{L, R\}^n \times [n]$. For every square permutation σ, the labels of (X, Y) are determined by the record types (the sequence X records if a point is a maximum (U) or a minimum (D) and the sequence

Y records if a point is a left-to-right record (L) or a right-to-left record (R)) and the anchor z_0 is determined by the value $\sigma^{-1}(1)$ (see Fig. 2 for an example).

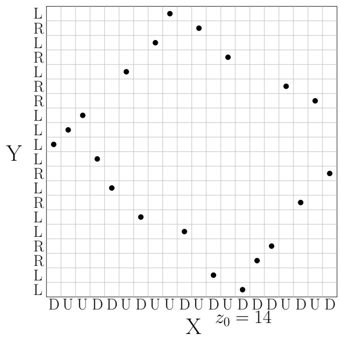

Figure 2: A square permutation σ with the associated anchored pair of sequences (X, Y, z_0). The sequence X is reported under the diagram of the permutation and the sequence Y on the left.

This projection map is not surjective, but we can identify subsets of anchored pairs of sequences (called *regular*) and of square permutations where the projection map is a bijection. We then construct a simple algorithm to produce a square permutation from regular anchored pairs of sequences. We show that asymptotically almost all square permutations can be constructed from regular anchored pairs of sequences, thus a permutation sampled uniformly from the set of regular anchored pairs of sequences will produce, asymptotically, a uniform square permutation.

Permuton limits, fluctuations and local limits

The first result we proved is the existence of the permuton limit for uniform square permutations. A permuton is a probability measure on the square $[0,1]^2$ with uniform marginals. Every permutation can be associated with the permuton induced by the sum of Dirac measures on points of the diagram of the permutation scaled to fit within $[0,1]^2$. We show that for a large square permutation σ that projects to a regular anchored pair of sequences, the permuton associated with σ is close to a permuton given by a rectangle (see for instance Fig. 1) embedded in $[0,1]^2$ with sides of slope ± 1 and bottom corner at $(\sigma^{-1}(1)/n, 0)$. This allows us to show that the permuton limit of uniform square permutations is a rectangle embedded in $[0,1]^2$ with sides of slope ± 1 and bottom corner at $(\mathbf{z}, 0)$, where $\mathbf{z}$ is a uniform point in the interval $[0,1]$ (we denote random quantities using **bold** characters).

The second result deals with fluctuations about the lines of the rectangle of the permuton limit. We show that they can be described by certain coupled Brownian motions. The latter arises naturally from the projection map described above, namely, the fluctuations around the bottom left edge of the rectangle (see Fig. 2) are determined by the distribution of the D's in the left part of the sequence X and of the L's in the lower part of the sequence Y. The coupling between Brownian motions comes from the fact that the total number of labels of each type on a given interval (either horizontal or vertical) sums up to the size of the interval.

The third result is a local limit theorem for square permutations. Our result is stated in terms of the local topology introduced by the speaker in [2]. We look at the neighborhood of a random element of a uniform square permutation and we study, for all $h \in \mathbb{N}$, the consecutive pattern induced by the h elements on the right and on the left of the chosen element, showing that, when the size of the whole permutation tends to infinity, this consecutive pattern converges in distribution to a random limiting pattern. Square permutations are the first natural but non-trivial model were the local limiting object is random (we recall that this is not the case for uniform random permutations avoiding a pattern of length three [2] or for uniform permutations in substitution-closed classes [3], where the local limiting objects are deterministic).

Our first and third results, *i.e.,* the permuton and local limits, can be interpreted in terms of the convergence of the proportion of occurrences and consecutive occurrences of any given pattern in a uniform random square permutation. We denote with $\widetilde{\text{occ}}(\pi,\sigma)$ (resp. $(\widetilde{\text{c-occ}}(\pi,\sigma))$ the proportion of occurrences (resp. consecutive occurrences) of a pattern π in σ, and with $\mathcal{S}$ the set of permutations. We can deduce that if $\boldsymbol{\sigma}_n$ is a uniform random square permutation of size n, then the following convergences (w.r.t. the product topology) hold:

$$(\widetilde{\text{occ}}(\pi,\boldsymbol{\sigma}_n))_{\pi\in\mathcal{S}} \xrightarrow{d} (\boldsymbol{\Lambda}_\pi)_{\pi\in\mathcal{S}} \quad \text{and} \quad \left(\widetilde{\text{c-occ}}(\pi,\boldsymbol{\sigma}_n)\right)_{\pi\in\mathcal{S}} \xrightarrow{d} (\boldsymbol{\Delta}_\pi)_{\pi\in\mathcal{S}},$$

where $(\boldsymbol{\Lambda}_\pi)_{\pi\in\mathcal{S}}$ and $(\boldsymbol{\Delta}_\pi)_{\pi\in\mathcal{S}}$ are random vectors that can be described in terms of the permuton and local limits of square permutations.

References

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