APPLICATIONS OF AZUMA'S INEQUALITY

We already saw a first application to the longest common substring problem. In this notes we explore two additional examples.

- **CHROMATIC NUMBER FOR THE ERDÖS - RENYI GRAPH**

$$\text{Let } \quad Z_n = \underset{\substack{\uparrow \\ \text{Chromatic number}}}{\chi}(\underset{\text{Erdös-Rényi graph}}{G_{n,p}}).$$

We recall that $G_{n,p}$ can be defined by the random variables $\{X_{i,j} \mid 1 \leq i < j \leq n\}$ where each $X_{i,j}$ is equal to 1 with prob. p (this means that there is an edge between the vertices i and j) and is equal to 0 with prob. $1-p$ (this means that there isn't an edge between the vertices i and j).

Let X_i denote the collection $\{X_{i,j}\}_{j>i}$.

We set

$$F_i := \mathbb{E}\left[Z_n \mid X_1, \dots, X_i\right], \quad \forall i \in [n].$$

Let $D_i = F_i - F_{i-1}$. Note that $\sum_{i=1}^{n} D_i = Z_n - \mathbb{E}[Z_n]$.

Since all the random variables $\{X_i\}_{i=1}^{n}$ are independent (because the edges of $G_{n,p}$ are independent) then the family $(D_i)_{1 \leq i \leq n}$ is a multiplicative system.

Moreover, changing any X_i changes the value of Z_n by at most 1 (only the color of the vertex i can be affected, why?)

Thus, we have $|D_i| \leq 1$ a.s. From Azuma's inequality we get that, for any $t_n > 0$,

$$\mathbb{P}\left(\left|Z_n - \mathbb{E}[Z_n]\right| \geq t_n\right) \leq 2 \cdot \exp\left(-\frac{t_n^2}{2\,n}\right).$$

NUMBER OF TRIANGLES IN THE ERDÖS-RÉNYI GRAPH

Let $T_n = \#$ of triangles in $G_{n,p}$. Erdös-Rényi graph

We fix an order on the set of edges $\{X_{i,j} \mid 1 \leq i < j \leq n\}$ of $G_{n,p}$ and we set

$$Y_i := i\text{-th edge of } G_{n,p}, \quad \forall i \in \left[\binom{n}{2}\right].$$

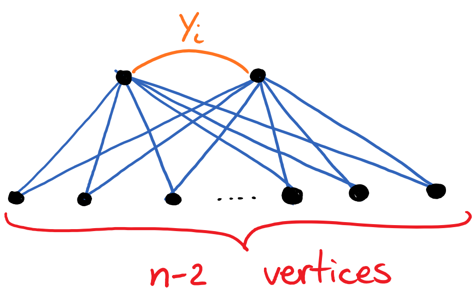
We define

$$F_i := \mathbb{E}\left[T_n \mid Y_1, \dots, Y_i\right], \quad \forall i \in \left[\binom{n}{2}\right].$$

Let $D_i = F_i - F_{i-1}$. Note that $\sum_{i=1}^{\binom{n}{2}} D_i = T_n - \mathbb{E}[T_n]$.

Since all the random variables $\{Y_i\}_{i=1}^{\binom{n}{2}}$ are independent (because the edges of $G_{n,p}$ are independent) then the family $(D_i)_{1 \leq i \leq 2n}$ is a multiplicative system.

Moreover, changing any Y_i changes the value of T_n by at most $n-2$. Indeed:



Adding / removing Y_i I can create / remove at most $(n-2)$ tringles!

Thus we have $|D_i| \leq n-2$, a.s. [Note that this is an example where the bound of the D_is depends on n]

From Azuma's inequality we get that, for all $s_n > 0$,

$$\mathbb{P}\left(\left| T_n - \mathbb{E}[T_n] \right| \geq s_n \right) \leq 2 \exp\left(- \frac{s_n^2}{2 \binom{n}{2} (n-2)^2} \right) \leq 2 \exp\left(- \frac{s_n^2}{n^4} \right).$$