Local limits for permutations:

an approach using generating trees & random walks

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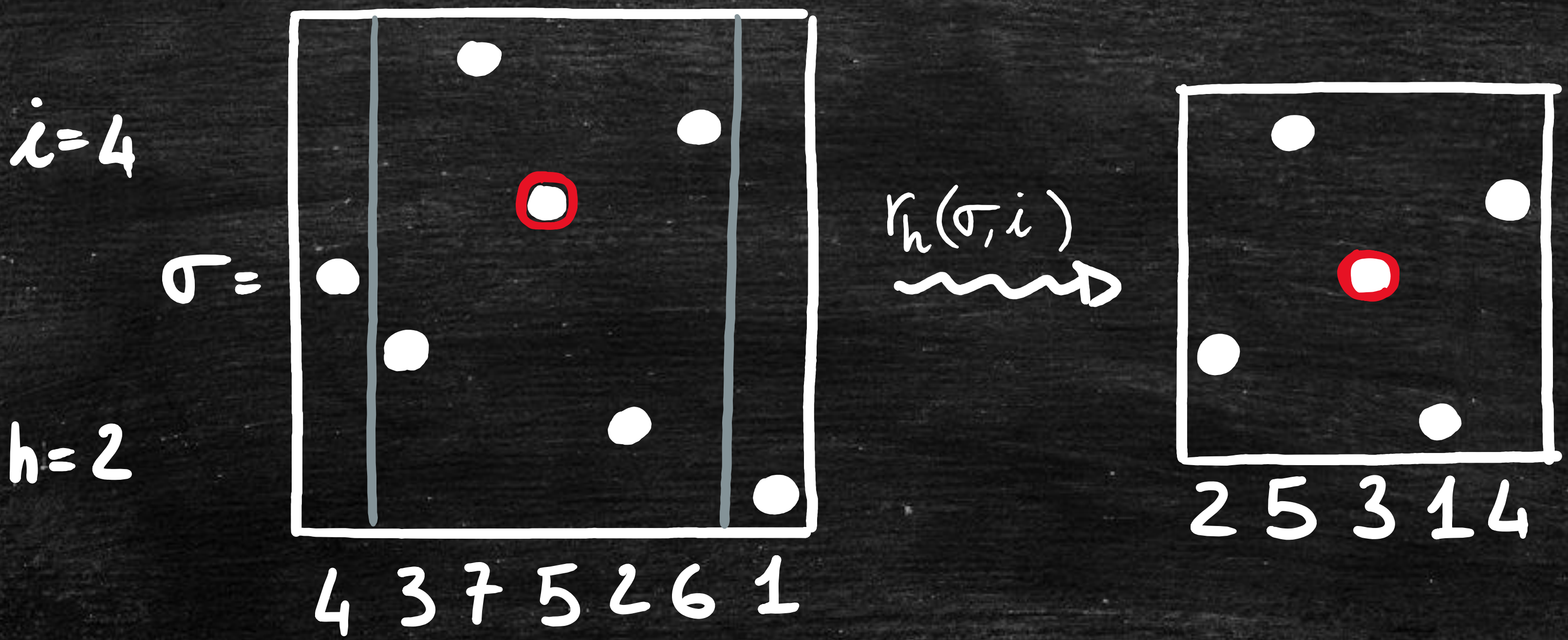
LOCAL TOPOLOGY FOR PERMUTATIONS



For a sequence of deterministic permutations $(\sigma^n)_{n\in\mathbb{N}}$

$$\sigma^n \xrightarrow{BS} \sigma^\infty_\bullet \iff \forall h\in\mathbb{N},\ r_h(\sigma^n, i_n) \xrightarrow{d} r_h(\sigma^\infty_\bullet)$$

↳ "infinite rooted permutations"

$$\sigma^n \xrightarrow{BS} \sigma_\bullet^\infty \iff \forall h \in \mathbb{N},\ r_h(\sigma^n, i_n) \xrightarrow{d} r_h(\sigma_\bullet^\infty)$$

For a sequence of random permutations $(\sigma^n)_{n\in\mathbb{N}}$

$$\sigma^n \xrightarrow{aBS} \sigma_\bullet^\infty \iff r_h(\sigma^n, i_n) \xrightarrow{d} r_h(\sigma_\bullet^\infty)$$

$$\forall h \in \mathbb{N}$$

$$\Updownarrow$$

$$\mathbb{E}\left[\widetilde{c\text{-}occ}(\pi, \sigma^n)\right] \longrightarrow \mathbb{E}\left[\widetilde{c\text{-}occ}(\pi, \sigma_\bullet^\infty)\right]$$

$$\forall \pi \in \mathcal{S}$$

$$\left(\widetilde{c\text{-}occ}(\pi, \sigma^n)\right)_{\pi\in\mathcal{S}} \xrightarrow{d} \left(\widetilde{c\text{-}occ}(\pi, \mu_\bullet^\infty)\right)_{\pi\in\mathcal{S}}$$

$$\Updownarrow$$

$$\sigma^n \xrightarrow{qBS} \mu_\bullet^\infty \iff \left(r_h(\sigma^n, i_n) \middle| \sigma^n\right) \xrightarrow{d} r_h(\mu_\bullet^\infty)$$

$$\forall h \in \mathbb{N}$$

Some results:

* Permutations avoiding one pattern of length 3 [B. '18]

  $Av(123)$ $Av(231)$

* Substitution-closed classes [B., Bouvel, Féray, Stufler '19]

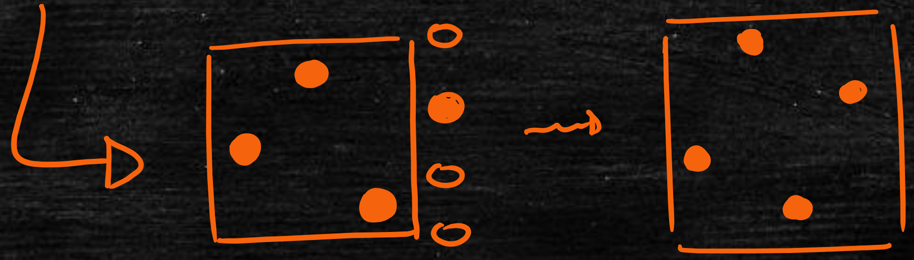
  $Av(2413, 3142), \ldots$

* Square permutations [B., Slivken '19]

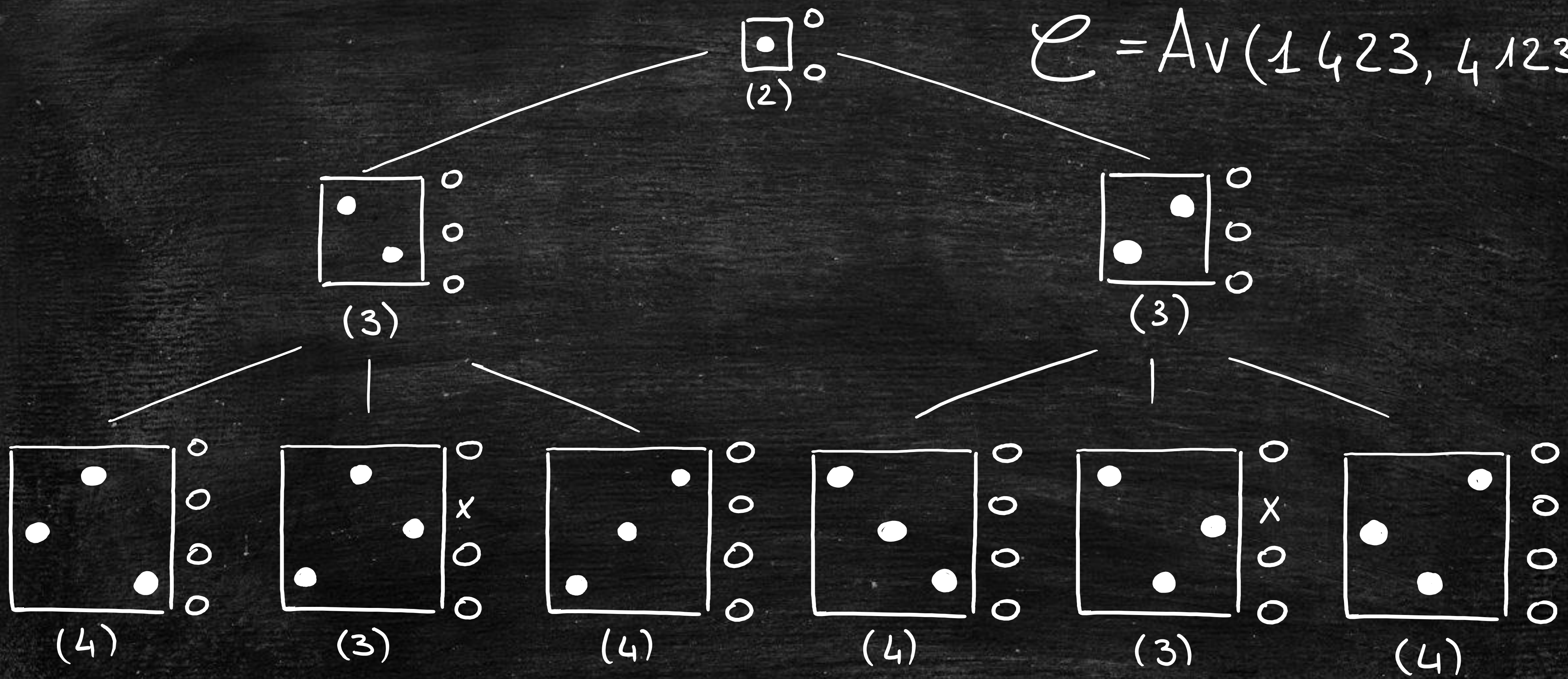
  $Av(\underbrace{24351, \ldots}_{\text{16 patterns of length 5}})$

Generating trees for permutations

The **generating tree** for a family of permutations $\mathcal{C}$ is the infinite rooted tree whose vertices are the permutations of $\mathcal{C}$ (each appearing exactly once in the tree) and such that permutations of size n are at level n. The children of some permutation $\sigma \in \mathcal{C}$ are obtained by appending a new final value to σ.

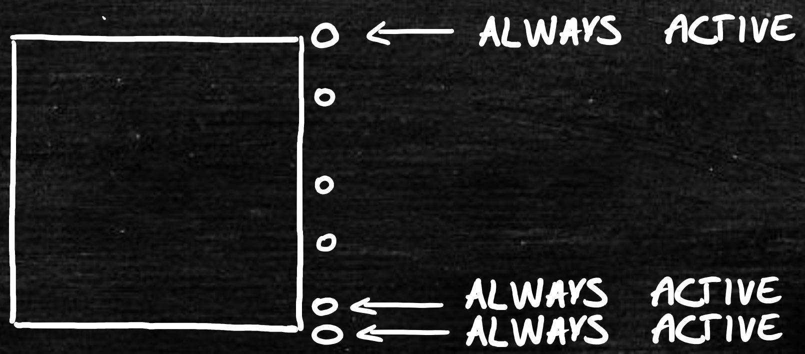


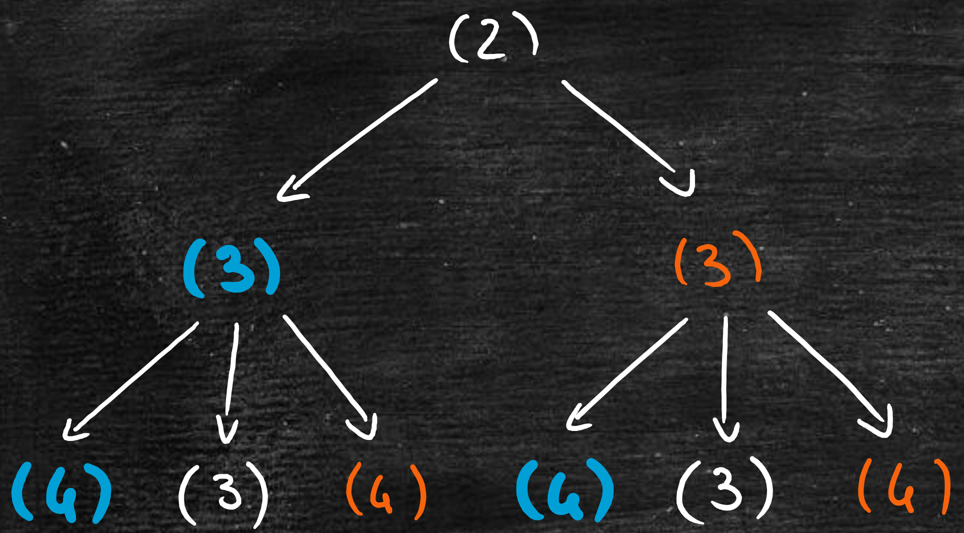
$\mathcal{C} = Av(1423, 4123)$



SUCCESSION RULE:

$$\begin{cases} \text{Root label: } (2) \\ (k) \longrightarrow (k+1)(3)(4)\cdots(k+1) \end{cases}$$



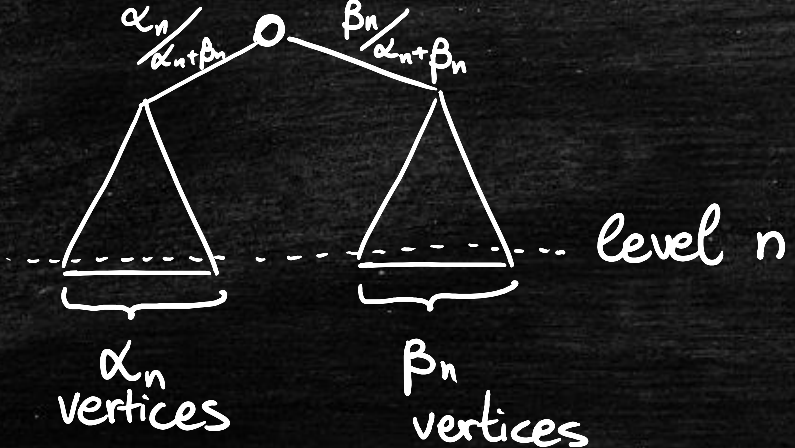


Bijection:

$Av^n(1423, 4123) \overset{H}{\longleftrightarrow}$ Colored paths in the generating tree of size n

In order to sample a uniform permutations of size n it is enough to sample a uniform path in the generating tree of size n.

1^{st} approach :



PROBLEM:

The distribution of the jumps depends on n !

BOLTZMAN SAMPLER

We consider the following bivariate generating function:

$$G(z,x) = \sum_{n \geq 0} \sum_{k \geq 0} g_{n,k} \, z^n x^k,$$

where $g_{n,k}$:= # of downwords path of size n starting at a vertex labelled by k

Notation: $\gamma_k(z) := \sum_{n \geq 0} g_{n,k} \, z^n$ $\qquad$ $F(z) := \gamma_1(z)$ with radius of convergence ρ

We consider the random walk $(X_i)_i$ on $\mathcal{X} \cup \{+\}$: $\longrightarrow$ absorbing state

- $\mathbb{P}(X_1 = \lambda) = 1$.
- $\mathbb{P}(X_{i+1} = \kappa' \mid X_i = \kappa) = r \cdot \mathrm{mult}_\kappa(\kappa') \dfrac{\gamma_{\kappa'}(r)}{\gamma_\kappa(r)}$, $\mathbb{P}(X_{i+1} = + \mid X_i = \kappa) = \dfrac{r}{\gamma_\kappa(r)}$.

Finally, $(\tilde{X}_i)_i$ is the r.w. with same distribution of $(X_i)_i$ where for every step from κ to κ' s.t. $\mathrm{mult}_\kappa(\kappa') \geq 2$ we color the label κ_i with a uniform color in $[\mathrm{mult}_\kappa(\kappa')]$.

$$S_n := H\big((\tilde{X}_i)_i \mid \underbrace{T_+}_{\text{first hitting time of } +} = n+1\big)$$

PROPOSITION: The permutation S_n is uniform in $\mathcal{C}_n$.

ASSUMPTIONS:

I - The generating tree has 1-dimensional labels, i.e. $\mathcal{L} \subseteq \mathbb{Z}$;

II - The power series F has positive radius of convergence ρ and $F(\rho) < \infty$;

III - There exists a probability distribution $(\alpha_y)_{y \in \mathbb{Z}_{\leq 1}}$ s.t.

$$\mathbb{P}\left(X_{i+1} = k+y \mid X_i = k \right) = \alpha_y \quad \text{whenever} \quad (k, k+y) \in \mathcal{L}^2;$$

IV - Combinatorial technical assumption.

"We can read the consecutive patterns from the jumps of the random walk"

THEOREM: Let $\mathcal{C}$ be a family of permutations s.t. the corresponding generating tree satisfies assumptions I, II, III and IV. Let $(\bar{Y}_i^*)_i$ be iid r.v. distributed as

$$\mathbb{P}\left(\bar{Y}_i^* = y\right) = \alpha_y \quad , \text{ for all } y \in \mathbb{Z}.$$

Let σ^n be a uniform random permutation in $\mathcal{C}_n$, then

$$\widetilde{c\text{-}occ}(\pi, \sigma^n) \xrightarrow{P} \mathbb{P}\left(\mathrm{Pat}(\bar{Y}_1^*, \dots, \bar{Y}_{|\pi|}^*) = \pi\right)$$

for all patterns $\pi \in \mathcal{S}$. Therefore, $\sigma^n \xrightarrow{qBS} \mathrm{Law}(\sigma_{\mathcal{C}}^\infty)$.

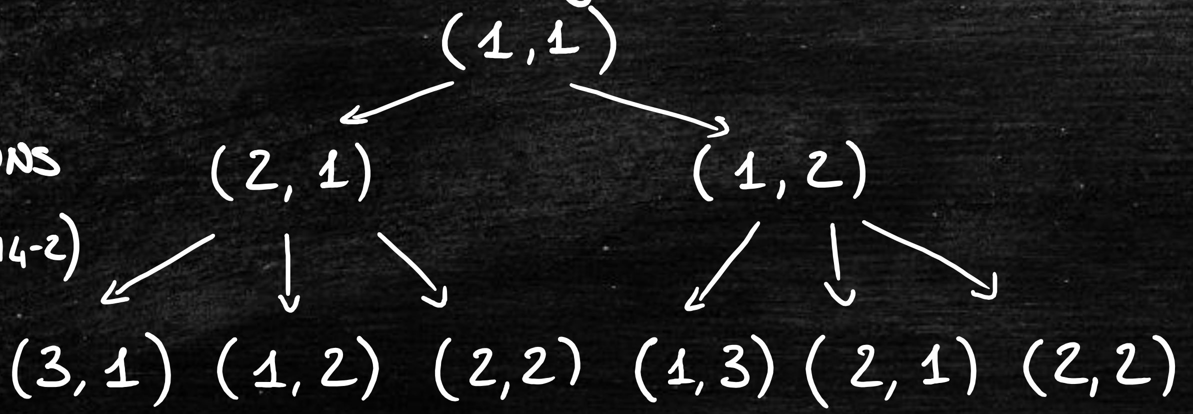
NEXT STEP?

- What about multi-dimensional labels?
- Can this "generating trees & random walks" approach help in understanding the PERMUTON limit?

ANSWER: joint work with MICKAËL MAAZOUN

BAXTER PERMUTATIONS

Av(2-41-3, 3-14-2)



Succession rule:

$$\begin{cases} \text{Root label: } (1,1) \\ (h,k) \rightsquigarrow \begin{matrix} (1,k+1),\ldots,(h,k+1) \\ (h+1,1),\ldots,(h+1,k) \end{matrix} \end{cases}$$

PROPOSITION:

Taking a uniform random Baxter permutation of size n, the associated random walk in the Baxter generating tree has the distribuzion of random walk with random initial starting point, i.i.d. steps, conditioned to first hit $(1,1)$ at time n and conditioned to stay in the positive quadrant.

- STUDY OF RANDOM WALKS IN CONES
- NEW TECHNIQUE FOR READING PATTERNS FROM THE WALK

$\Downarrow$

NEW RANDOM PERMUTON LIMIT

THANK YOU!