Exercise 1

1) $$X_n = \sum_{i=1}^{n-1} \mathbb{1}_{\{\sigma(i) > \sigma(i+1)\}}$$

2) $$\kappa_r(X_n) = \kappa(\underbrace{X_n, \dots, X_n}_{r\text{-times}}) = \sum_{i=1}^{n-1} \kappa(\mathbb{1}_{\{\sigma(i)>\sigma(i+1)\}}, \underbrace{X_n, \dots, X_n}_{(r-1)\text{-times}})$$
$$= \sum_{i_1,\dots,i_r=1}^{n-1} \kappa(\mathbb{1}_{\{\sigma(i_1)>\sigma(i_1+1)\}}, \dots, \mathbb{1}_{\{\sigma(i_r)>\sigma(i_r+1)\}})$$

3) Consider a tuple $(i_1, \dots, i_r) \in [n-1]^r$ and w.l.o.g. assume that $i_1 < i_2 < \dots < i_r$. Note that if $\exists\ j \in [r-1]$ s.t. $i_j < i_{j+1} - 1$ (i.e. $i_{j+1} \neq i_j$ and $i_{j+1} \neq i_j + 1$) then

$$\{\mathbb{1}_{\{\sigma(i_s)>\sigma(i_s+1)\}}\}_{s=1}^{j} \perp\!\!\!\perp \{\mathbb{1}_{\{\sigma(i_\ell)>\sigma(i_\ell+1)\}}\}_{\ell=j+1}^{r}$$

and so

$$\kappa(\mathbb{1}_{\{\sigma(i_1)>\sigma(i_1+1)\}}, \dots, \mathbb{1}_{\{\sigma(i_r)>\sigma(i_r+1)\}}) = 0.$$

4. Therefore $\kappa(\mathbb{1}_{\{\sigma(i_1)>\sigma(i_1+1)\}}, \dots, \mathbb{1}_{\{\sigma(i_r)>\sigma(i_r+1)\}})$ is different from zero iff $\exists$ an order of $\{i_j\}_{j=1}^r$ s.t. for all $j \in [r-1]$, either $i_{j+1} = i_j$ or $i_{j+1} = i_j + 1$. The number of such tuples $(i_1, \dots, i_r) \in [n-1]^r$ is bounded by

possible reorderings.

$$n \cdot 2^{r-1} \cdot r! = O(n).$$

choices for the first element

choices for the other $(r-1)^{th}$ elements

5. Set $\hat{X}_n = \dfrac{X_n - \mathbb{E}[X_n]}{\quad}$ Then if $r > 2$.

5. Set $\hat{X}_n = \frac{X_n - \mathbb{E}[X_n]}{\sqrt{Var(X_n)}}$. Then, if $r > 2$,

lemma 1 — point 4

$$K_r(\hat{X}_n) = \frac{K_r(X_n)}{Var(X_n)^{r/2}} \leq \frac{B_r \cdot O(n)}{\left(\frac{n}{12} + O(1)\right)^{r/2}} \xrightarrow{n\to\infty} 0$$

point 3 (most of the terms are zero)

Obviously $K(\hat{X}_n) \to 0$ and $K_2(\hat{X}_n) \to 1$ therefore, using the moment method via cumulants we can conclude that

$$\hat{X}_n \xrightarrow{d} N(0,1).$$

Exercise 2

1) $Z(x) := \sum_{p \leq n^{0.1}} \mathbb{1}_{\{p|x\}}$. We want to show that $|\nu(x) - Z(x)| \leq 10$.

Assume by contradiction that $|\nu(x) - Z(x)| > 10$, then there exist $n \geq p_1, \dots, p_{11} \geq n^{0.1}$ prime numbers s.t.

$$x = p_1 \cdot p_2 \cdot \ldots \cdot p_{11} \cdot k \quad \text{with} \quad k \geq 1$$

and so $x \geq (n^{0.1})^{11} \cdot k > n$, obtaining a contradiction.

2) $\mathbb{E}[\mathbb{1}_{\{p|x\}}] = \mathbb{P}(p \mid x) = \mathbb{P}\left(x = p \cdot k \text{ with } 1 \leq k \leq \lfloor \frac{n}{p} \rfloor\right) = \frac{\lfloor n/p \rfloor}{n}$.

3) $\mathbb{E}[Z(x)] = \mathbb{E}\left[\sum_{p \leq n^{0.1}} \mathbb{1}_{\{p|x\}}\right] = \sum_{p \leq n^{0.1}} \mathbb{E}[\mathbb{1}_{\{p|x\}}] = \sum_{p \leq n^{0.1}} \frac{\lfloor n/p \rfloor}{n}$

point 2)

$$= \sum_{p | n^{0.1}} \frac{1}{p} + \underbrace{\sum_{p | n^{0.1}} \varepsilon_{n,p}}_{= O(1)} = \log(\log(n)) + O(1).$$

$$= \underbrace{\sum_{p \mid n^{0.1}} \frac{1}{p}}_{= \log(\log(n^{0.1})) = \log\log(n) + \log(10)} + \overbrace{\sum_{p \mid n^{0.1}} \varepsilon_{n,p}}^{= O(1)} = \log(\log(n)) + O(1).$$

$$\frac{\lfloor \frac{n}{p} \rfloor}{n} = \frac{1}{p} \pm \varepsilon_{n,p} \quad \text{with} \quad |\varepsilon_{n,p}| \leq \frac{1}{p}$$

4. $$\operatorname{Var}[Z(x)] = \operatorname{Var}\left(\sum_{p \leq n^{0.1}} 1\!\!1_{\{p \mid x\}}\right) = \sum_{p \leq n^{0.1}} \underbrace{\operatorname{Var}\left(1\!\!1_{\{p \mid x\}}\right)}_{= \mathbb{E}[1\!\!1_{\{p \mid x\}}] - \mathbb{E}[1\!\!1_{\{p \mid x\}}]^2} + 2 \sum_{p < q \leq n^{0.1}} \operatorname{Cov}\left(1\!\!1_{\{p \mid x\}}, 1\!\!1_{\{q \mid x\}}\right)$$

Fix $p < q \leq n^{0.1}$

$$\operatorname{Cov}\left(1\!\!1_{\{p \mid x\}}, 1\!\!1_{\{q \mid x\}}\right) = \overset{\lfloor n/pq \rfloor / n}{\mathbb{E}\left[1\!\!1_{\{p \mid x \text{ and } q \mid x\}}\right]} - \overset{\lfloor n/p \rfloor / n}{\mathbb{E}\left[1\!\!1_{\{p \mid x\}}\right]} \cdot \overset{\lfloor n/q \rfloor / n}{\mathbb{E}\left[1\!\!1_{\{q \mid x\}}\right]}$$

$$= \left(\frac{1}{pq} \pm \varepsilon_{pq,n}\right) - \left(\frac{1}{p} \pm \varepsilon_{p,n}\right)\left(\frac{1}{q} \pm \varepsilon_{q,n}\right)$$

$$= \pm \varepsilon_{pq,n} \pm \frac{\varepsilon_{q,n}}{p} \pm \frac{\varepsilon_{p,n}}{q} \pm \varepsilon_{p,n}\varepsilon_{q,n} \leq \frac{4}{n}$$

Therefore

$$\operatorname{Var}(Z(x)) = \mathbb{E}[Z] + \underbrace{\sum_{p \leq n^{0.1}} \overset{\leq 1/p^2}{\mathbb{E}\left[1\!\!1_{\{p \mid x\}}\right]^2} + 2 \sum_{p < q \leq n^{0.1}} \overbrace{\operatorname{Cov}\left(1\!\!1_{\{p \mid x\}}, 1\!\!1_{\{q \mid x\}}\right)}^{\leq 4/n}}_{O(1)}$$

5. By Chebichev's inequality

$$\mathbb{P}\left(|Z - \mathbb{E}[Z]| \geq \varepsilon \sqrt{\operatorname{Var}(Z)}\right) \leq \frac{1}{\varepsilon^2}$$

Therefore, setting $\varepsilon = \omega'(n) \xrightarrow[n \to \infty]{} \infty$, for all but $o(n)$ integers $x \leq n$, we have

$$\left| Z - \left(\log(\log(n)) + O(1)\right) \right| \leq \omega'(n) \sqrt{\log\log(n) + O(1)}.$$

Therefore, for any slowly increasing sequence $\omega(n) \to \infty$, chosing an appropriate $\omega'(n)$ in the previous equation, we obtain that:

$$\left| Z - \log\log(n) \right| \leq \omega(n) \cdot \sqrt{\log\log(n)}, \text{ for all but } o(n) \text{ integers } x \leq n.$$