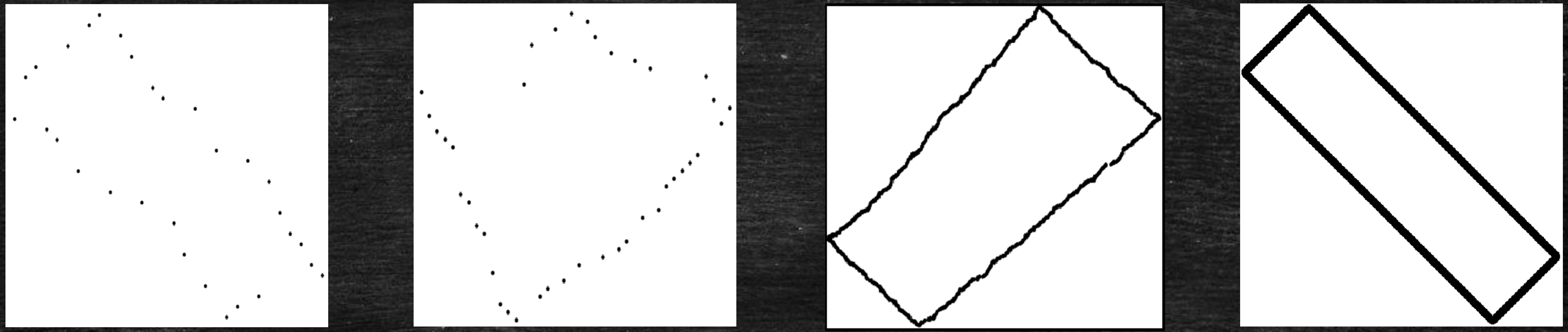


Square permutations are typically rectangular

(joint work with E. Slivken)

J. Borga, UZH

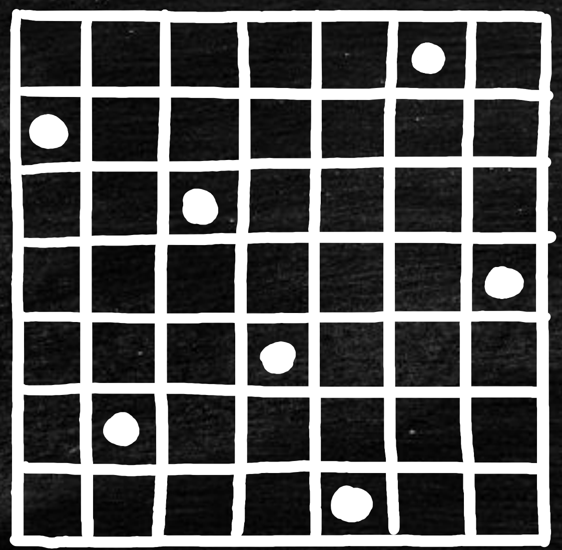
June 20th, 2019

A graphical point of view on permutations

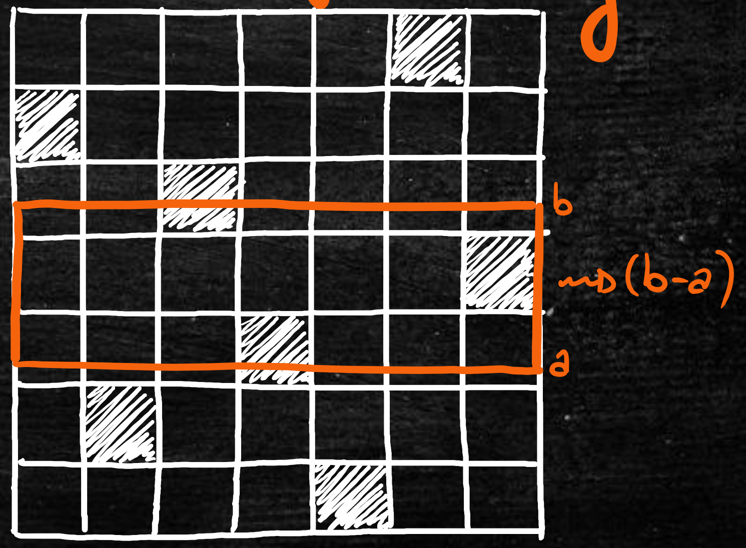
Consider the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 2 & 5 & 3 & 1 & 7 & 4 \end{pmatrix}$:

Probability measure on the unit-square with uniform marginals

$\sigma =$



$\rightsquigarrow$ $\mu_\sigma =$



Def: A **PERMUTON** is a probability measure on the square $[0,1]^2$ with uniform marginals.

Remark: We have a natural notion of convergence of such objects: the **WEAK CONVERGENCE**. This defines a nice compact space.

$\Longrightarrow$ limits of permutons are permutons, i.e., potential limits of sequences of permutations also have uniform marginals.

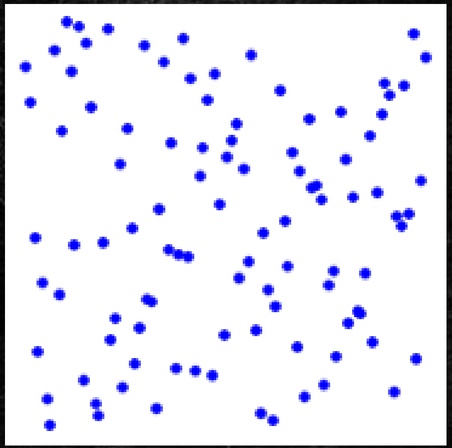
Moreover, every permuton is the limit of a sequence of μ_{σ_n}!

Example: uniform random permutations

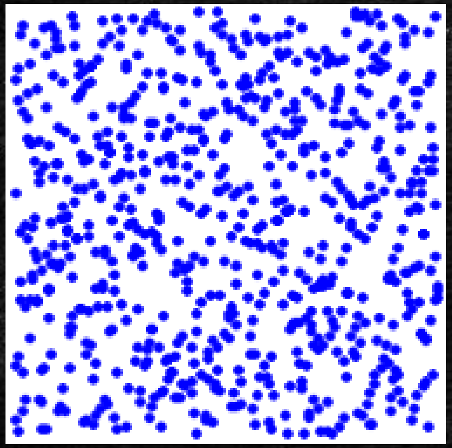
Let σ^n be a uniform permutation of size n. Then

$$\mu_{\sigma^n} \xrightarrow{d} \mathrm{Leb}([0,1]^2).$$

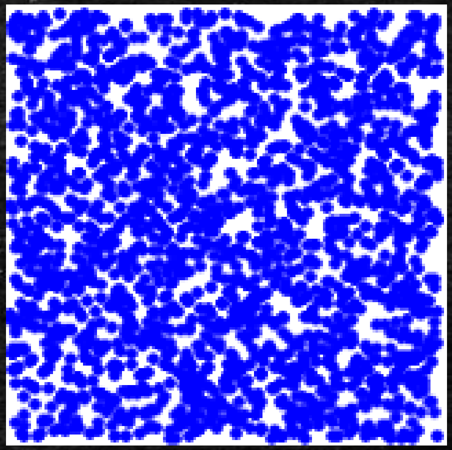
Proof by pictures:



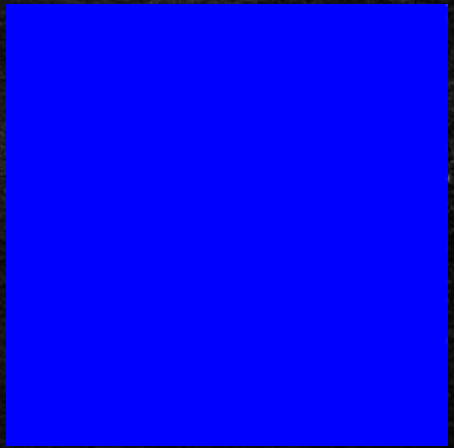
$n=100$



$n=700$



$n=2000$



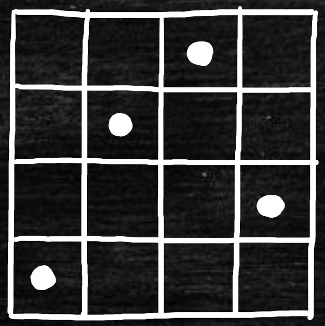
$\mathrm{Leb}([0,1]^2)$

Permutation patterns

Def: An occurrence of a pattern $\pi \in \mathcal{S}^k$ in $\sigma \in \mathcal{S}^n$ is a subsequence $\sigma_{i_1} \cdots \sigma_{i_k}$ that is order-isomorphic to π, i.e.,

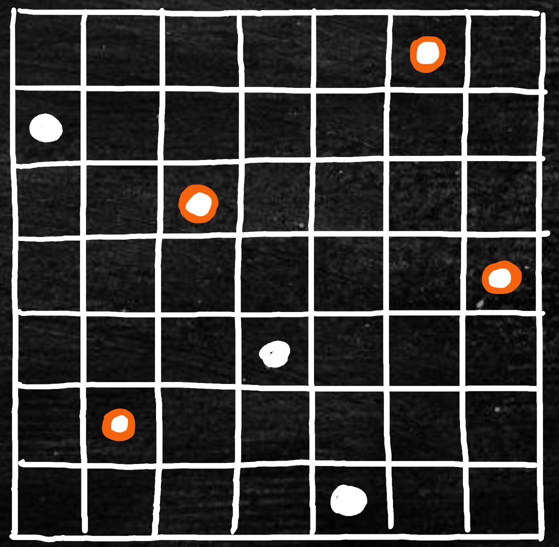
$$\sigma_{i_s} < \sigma_{i_t} \iff \pi_s < \pi_t .$$

Example: Occurrences of 1342



6 $\underline{2}$ $\underline{5}$ 3 1 $\underline{7}$ $\underline{4}$

1 3 4 2

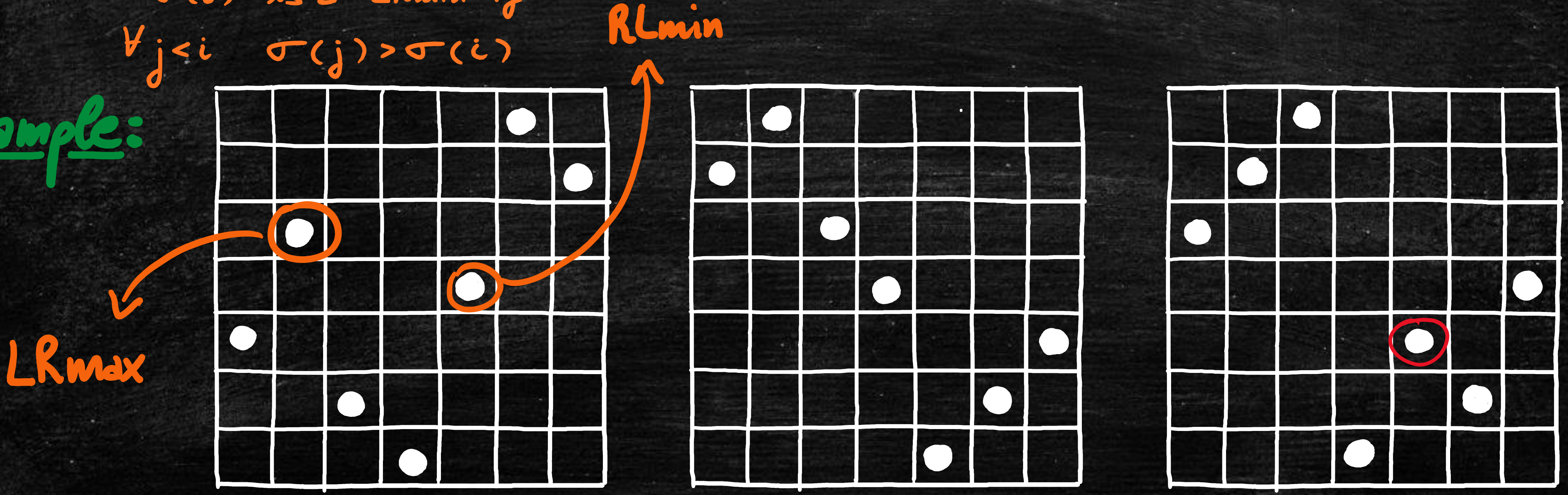


Square permutations

Def: A **square** permutation is a permutation where every point is a **RECORD**, i.e., a left-to-right minimum or maximum, or a right-to-left minimum or maximum.

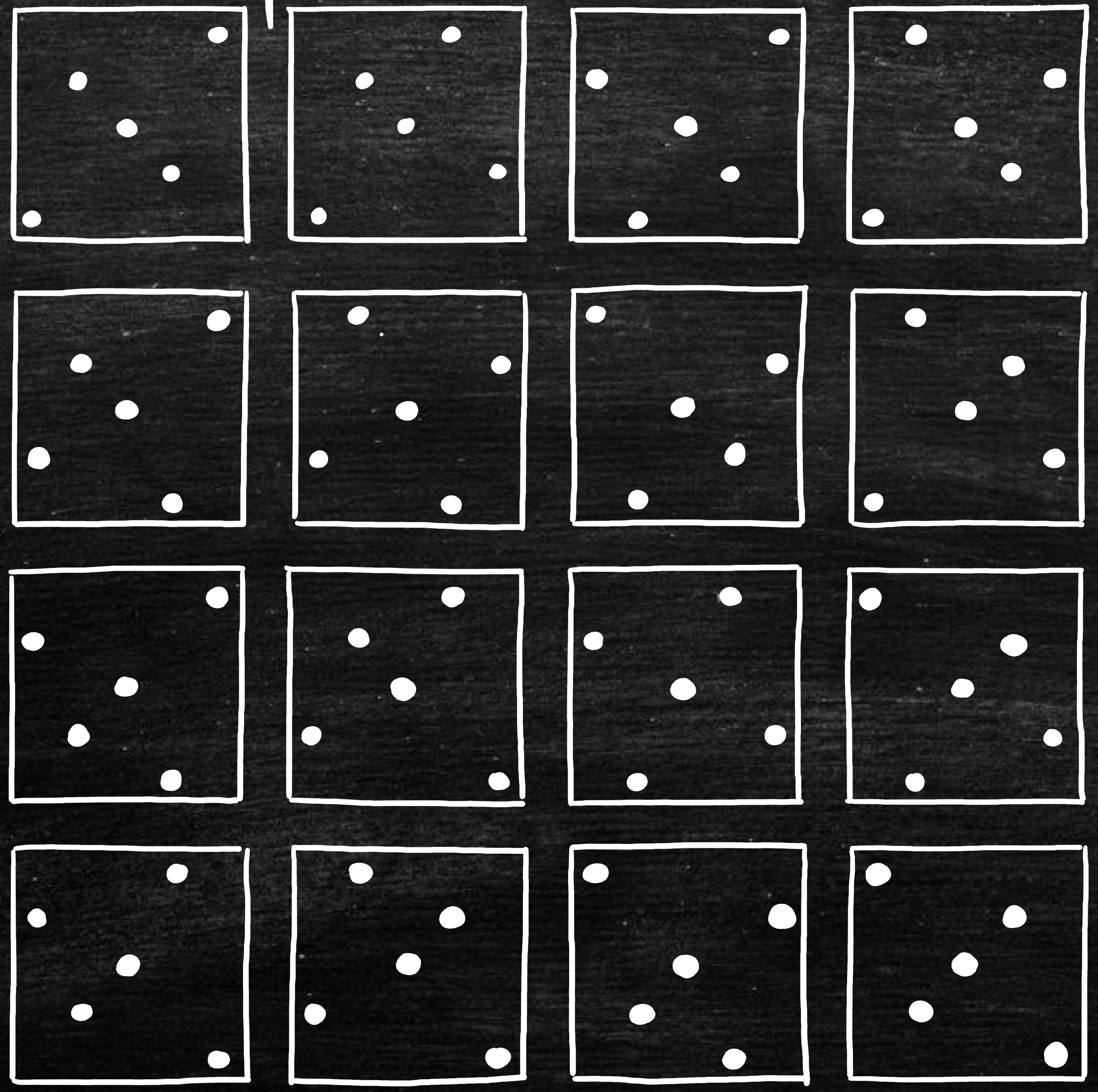
$\sigma(i)$ is a LRmin if $\forall j<i \quad \sigma(j)>\sigma(i)$

Example:



SQUARE PERMUTATIONS AS A PATTERN-AVOIDING CLASS:

Square permutations are permutations that avoid the following 16 patterns:



Sometimes they are also called:

CONVEX PERMUTATIONS

ENUMERATION:

Mansour & Severini (2007)

Duchi & Poulalhon (2008)

$$\#\text{ square permutations of size } n = \underbrace{2(n+2)\,4^{n-3} + \underbrace{4(2n-5)\binom{2n-6}{n-3}}_{o(n\,4^{n-3})}}_{\sim 2(n+2)\,4^{n-3}}$$

QUESTION:

How do we sample a uniform square permutation?

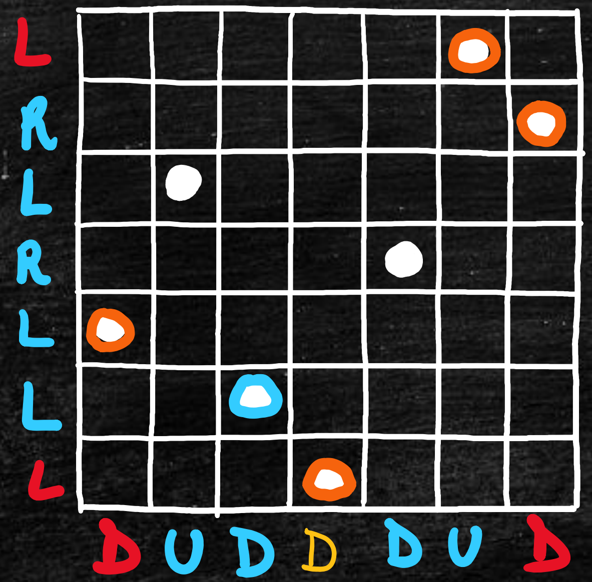
Sampling uniform square permutations

We consider the following projection map:

$$\varphi : Sq(n) \longrightarrow \{U,D\}^n \times \{L,R\}^n \times [n]$$

The space of anchored pairs of sequences of labels.

Y



$\longmapsto (X, Y, z_0)$

X $\hookrightarrow z_0 = 4$

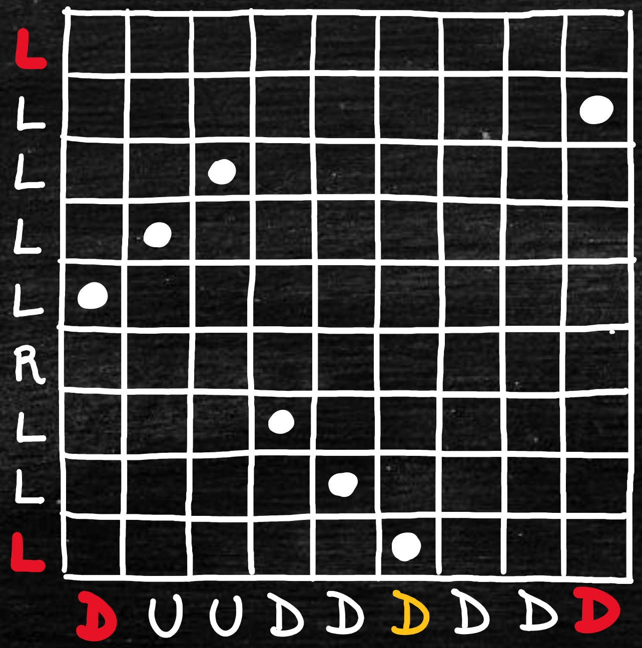
Def: We say that (X, Y, z_0) is a **good** anchored pair of sequences if

$$X_1 = X_n = X_{z_0} = D \quad \text{and} \quad Y_1 = Y_n = L.$$

Note that # of possible good anchored pairs $= 2^{n-2}\left(2 \cdot 2^{n-2} + (n-2)\,2^{n-3}\right) = 2(n+2)\,4^{n-3}.$

PROBLEM: We can not reconstruct a square permutation from all good anchored pairs of sequences.

Counterexample:



← FAIL ☹

↳ Why?

TOO MANY Ls & Ds!

We need some regularity conditions on our sequences that are satisfied by "asymptotically almost all" good anchored pairs.

Notation:

- $ct_D(i)$ = # of Ds in X up to (and including) position i
- $pos_D(i)$ = index of the i-th D in X.

PETROV CONDITIONS:

(1) $\left|ct_D(i) - ct_D(j) - \frac{1}{2}(i-j)\right| < n^{.4}$, for $|i-j| < n^{.6}$;

(2) $\left|ct_D(i) - ct_D(j) - \frac{1}{2}(i-j)\right| < \frac{1}{2}|i-j|^{.6}$, for $|i-j| > n^{.3}$;

(3) $\left|pos_D(i) - pos_D(j) - 2(i-j)\right| < n^{.4}$, for $|i-j| < n^{.6}$;

(4) $\left|pos_D(i) - pos_D(j) - 2(i-j)\right| < 2|i-j|^{.6}$, for $|i-j| > n^{.3}$.

Def: We say that a good anchored pair (X, Y, z_0) is **regular** if:

- X and Y satisfy the Petrov conditions,
- $n^{q} \leqslant z_0 \leqslant n - n^{q}$

We denote by Ω_n the space of regular anchored pairs of size n.

Lemma 1: $\varphi^{-1} : \Omega_n \longrightarrow Sq(n)$ is well-defined and injective.

Lemma 2: Let (X, Y, z_0) be chosen independently and uniformly at random from $\{U, D\}^n \times \{L, R\}^n \times [n]$, then $\mathbb{P}\big((X, Y, z_0) \in \Omega_n\big) \geqslant 1 - o(1)$.

Theorem: With probability $1 - o(1)$ a uniform square permutation σ_n of size n belongs to $\varphi^{-1}(\Omega_n)$.

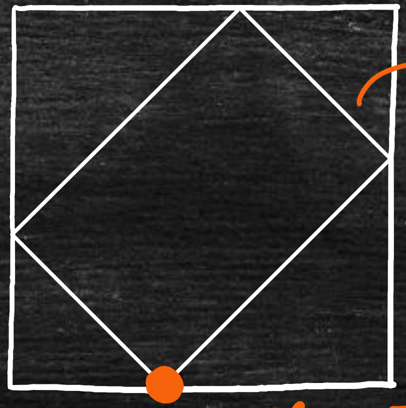
Proof:

$$\mathbb{P}\big(\sigma_n \in \varphi^{-1}(\Omega_n)\big) \underbrace{=}_{\varphi^{-1} \text{ is injective}} \frac{|\Omega_n|}{|Sq(n)|} = \frac{\overbrace{2(n+2)4^{n-3}}^{\#\text{ good anchored pairs}}\ \overbrace{(1 - o(1))}^{\text{Lemma 2}}}{\underbrace{2(n+2)4^{n-3}(1 - o(1))}_{\text{"enumerative result"}}} \longrightarrow 1. \quad \square$$

THEOREM: [SQUARE PERMUTATIONS ARE TYPICALLY RECTANGULAR]

Let σ_n be a uniform random square permutation of size n, then

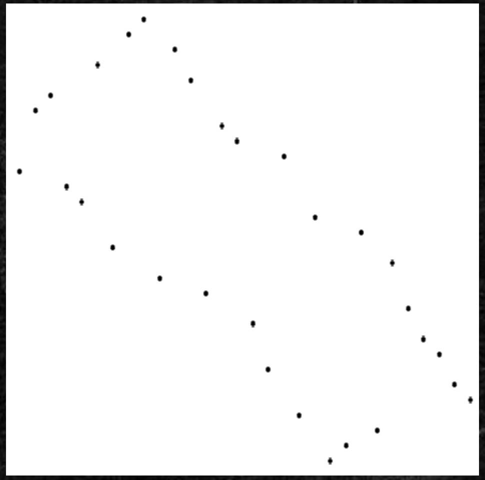
$$\mu_{\sigma_n} \xrightarrow{d} \mu^z =$$



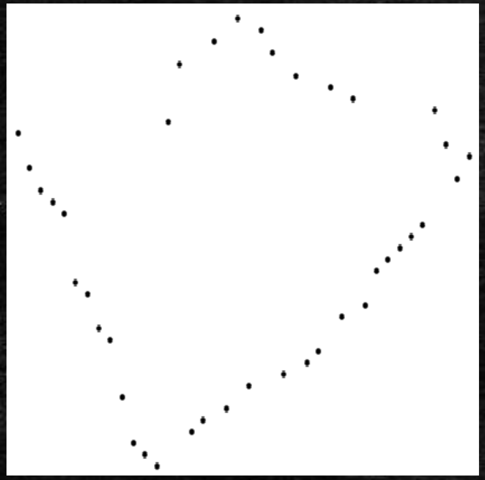
Lebesgue measure on the rectangle with total mass 1.

$z \sim \mathrm{Unif}([0,1])$

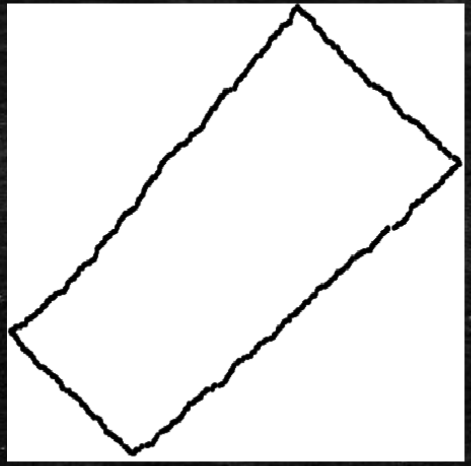
Simulations:



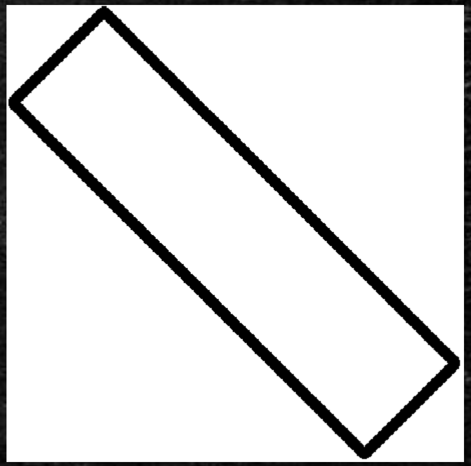
$n=30$



$n=40$

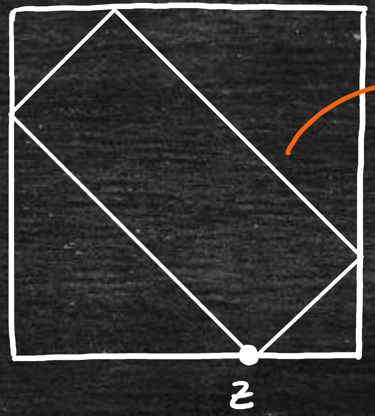


$n=1000$



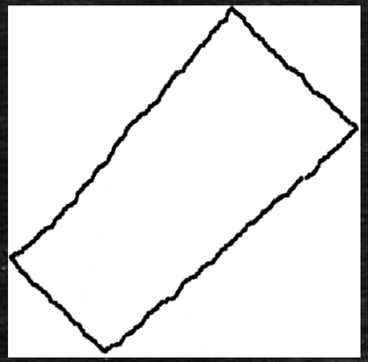
$n=1000000$

Proof: For every $z\in(0,1)$ we define the permuton

$\mu^z =$ 

Lebesgue measure on the rectangle with total mass 1

Then, for every square permutation $\sigma_n \in \varphi^{-1}(\Omega_n)$, we consider the

permuton $\mu_{\sigma^n} =$ 

We show that

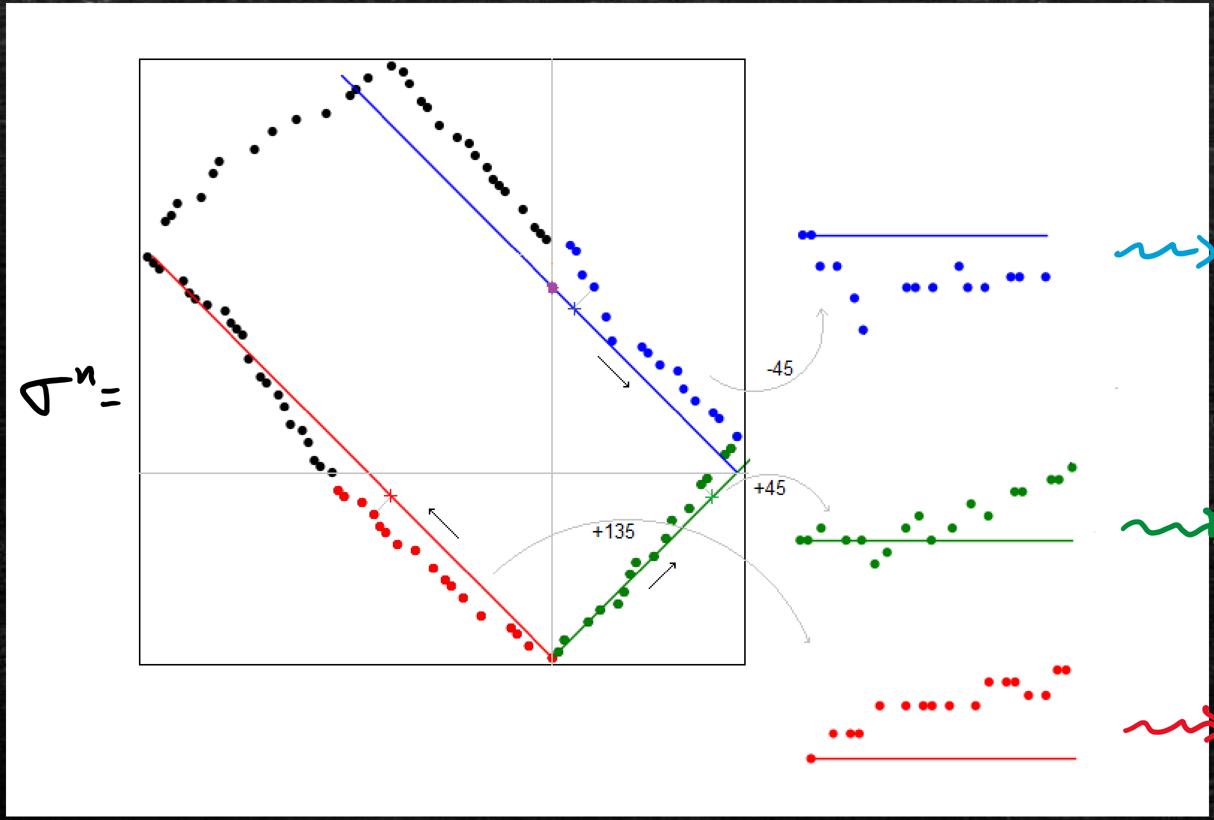
$$\sup_{\sigma^n\in\varphi^{-1}(\Omega_n)} d_{\square}\left(\mu_{\sigma^n}, \mu^{z_n}\right) < C\, n^{-.4}; \quad \text{where } z_n = \sigma^{-1}(1)/n.$$

↳ metric for the permuton topology.

Fluctuations

QUESTION: What happens if instead of a factor n we rescale distances by a factor $\sqrt{n}$?

$\sigma^n =$



$\rightsquigarrow \left(F^{\sigma^n}(t)\right)_{t\in[0,1]}$

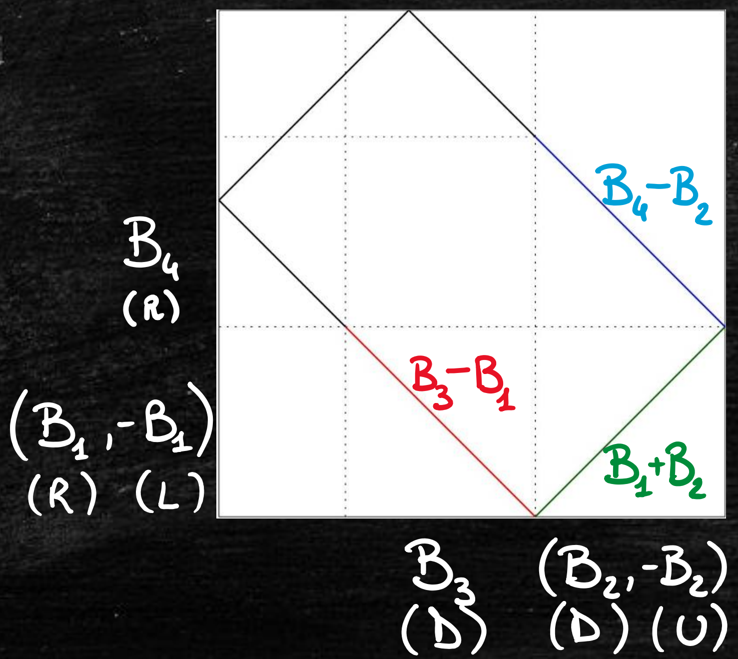
$\rightsquigarrow \left(F^{\sigma^n}(t)\right)_{t\in[0,1]}$

$\rightsquigarrow \left(F^{\sigma^n}(t)\right)_{t\in[0,1]}$

THEOREM: Let σ^n be a uniform random square permutation of size n, and $B_1(t), B_2(t), B_3(t), B_4(t)$ be four independent standard Brownian motions on the interval $[0,1]$. Conditioning on $z_0 = t_n$, with $\frac{n}{2} + Cn^{\delta} < t_n \leq n - n^{\gamma}$, we have the following convergence in distribution

$$\left(F^{\sigma^n}(t),\ F^{\sigma^n}(t),\ F^{\sigma^n}(t)\right)_{t\in[0,1]} \xrightarrow{d} \left(B_1(t)+B_2(t),\ B_3(t)-B_1(t),\ B_4(t)-B_2(t)\right)_{t\in[0,1]}.$$

Proof:



DIFFICULTIES:

- Families of points have random cardinalities
- The interpolating processes are random in both coordinates

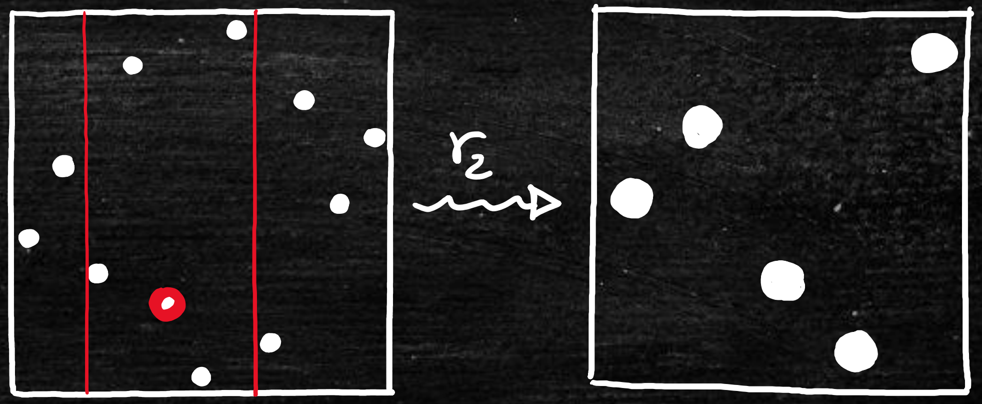
A LOCAL POINT OF VIEW

We can look at square permutations from a <u>local point of view</u>, that is, we look at the neighbourhoods of a <u>random</u> element of a uniformly <u>random</u> square permutations. We study, for all $h \in \mathbb{N}$, the consecutive pattern induced by the h elements on the right and on the left of the chosen element, i.e.

$$r_h(\sigma^n, i_n)$$

uniform square permutation

uniform index from 1 to n



$(\sigma^n, 5)$ $\qquad$ $r_2(\sigma^n; 5)$

The presence of two sources of randomness, one for the choice of the permutation and one for the choice of the root, leads to two NON-EQUIVALENT possible analysis:

① $\lim_{n\to\infty} \mathbb{P}(r_h(\sigma^n, i_n) = (\pi, h+1)) \quad \forall \pi \in \mathcal{S}^{2h+1}, h \geq 1$ [ANNEALED B-S CONVERGENCE]

② $\lim_{n\to\infty} \Big(\mathbb{P}\big(r_h(\sigma^n, i_n) = (\pi, h+1) \,\big|\, \sigma^n\big)\Big)_{\pi \in \mathcal{S}^{2h+1}, h \geq 1}$ [QUENCHED B-S CONVERGENCE]

THEOREM: Let σ^n be a uniform square permutation, then σ^n quenched B-S converges. for the topology defined in a previous work!

Remark: We have an explicit construction of the limiting object.

Connection with permutation patterns

Notation:

$$\widetilde{occ}(\pi,\sigma) = \frac{\#\text{ occurrences of } \pi \text{ in } \sigma}{\binom{|\sigma|}{|\pi|}}$$

$$\widetilde{c\text{-}occ}(\pi,\sigma) = \frac{\#\text{ consecutive occurrences of } \pi \text{ in } \sigma}{|\sigma|}$$

Corollary: Let σ^n be a uniform square permutations, then

$$\left(\widetilde{occ}(\pi,\sigma)\right)_{\pi\in\mathcal{S}} \xrightarrow{d} (\Lambda_\pi)_{\pi\in\mathcal{S}} \quad \& \quad \left(\widetilde{c\text{-}occ}(\pi,\sigma)\right)_{\pi\in\mathcal{S}} \xrightarrow{d} (\Delta_\pi)_{\pi\in\mathcal{S}}$$

where $\forall \pi\in\mathcal{S}$, Λ_π (resp. Δ_π) can be described in terms of the permuton (resp. local) limiting object.

WHAT ABOUT "ADDING" INTERNAL POINTS?

(WORK IN PROGRESS WITH DUCHI & SLIVKEN)

Notation: $ASq(n,k)$ = Permutations of size $n+k$ with k internal points

Theorem: Let $k = o(n^{1/2-\delta})$ for some $\delta > 0$, then

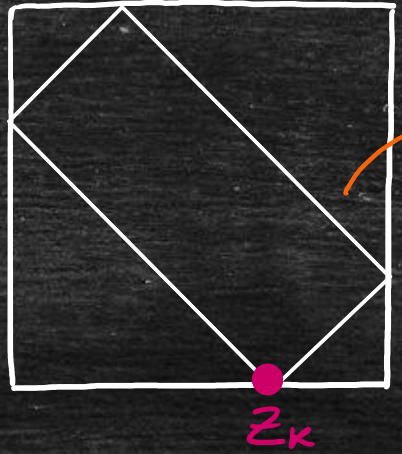
$$|ASq(n,k)| \underset{n\to\infty}{\sim} \frac{k!\, 2^{k+1}\, n^{2k+1}\, 4^{n-3}}{(2k+1)!}.$$

Q: What happens to the PERMUTON LIMIT?

Theorem: Fix $\kappa > 0$. Let σ_n be a uniform permutation in $Asq(n,\kappa)$,

then

$$\mu_{\sigma_n} \xrightarrow{d} \mu^{z_\kappa} =$$



Lebesgue measure on the rectangle with total mass 1

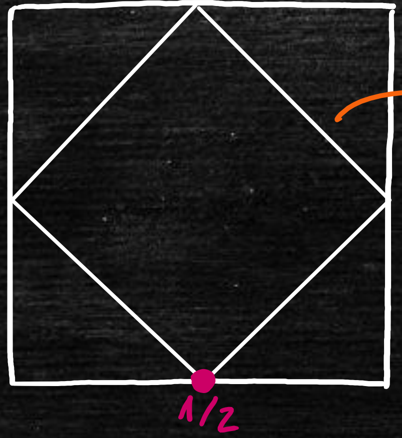
where

$$\mathbb{P}(z_\kappa < s) = (2\kappa+1)\binom{2\kappa}{\kappa}\int_0^s \big(t(1-t)\big)^{\kappa}\, dt \quad \forall s \in (0,1).$$

Moreover, if $\kappa \longrightarrow +\infty$ and $\kappa = o(n^{\frac{1}{2}-\delta})$ for some $\delta > 0$

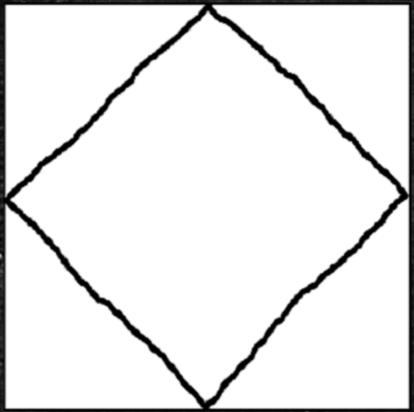
then

$$\mu_{\sigma_n} \xrightarrow{d} \mu^{1/2} =$$

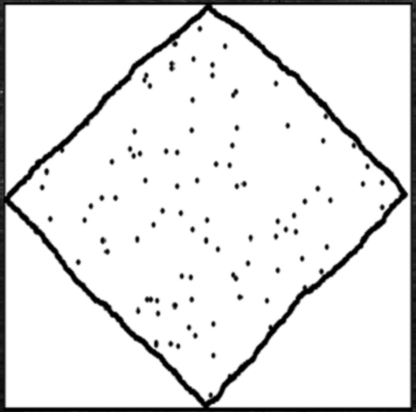


Lebesgue measure on the square with total mass 1

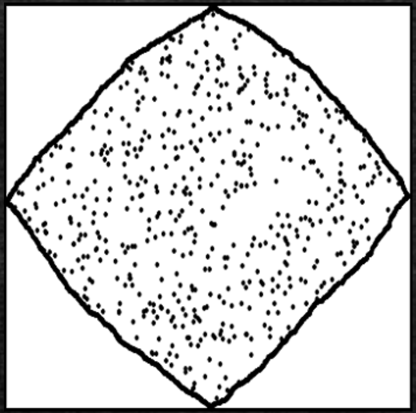
Next step?!



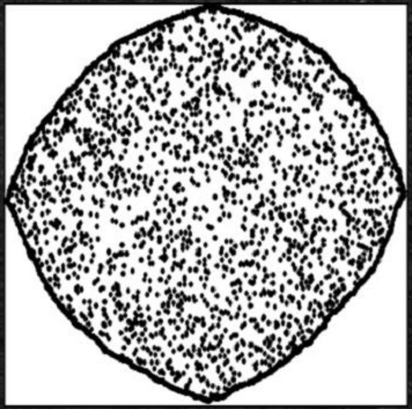
n=2000



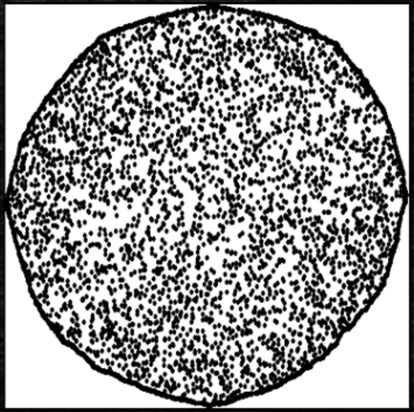
K=100



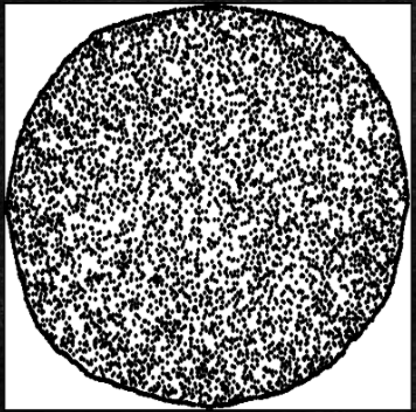
K=500



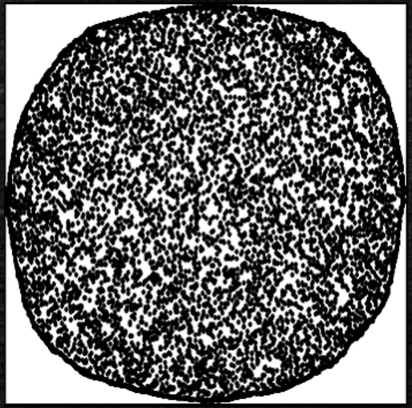
K=2000



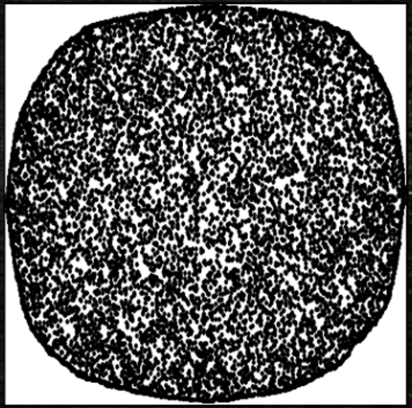
K=4000



K=6000



K=8000



K=10'000

THANK YOU!