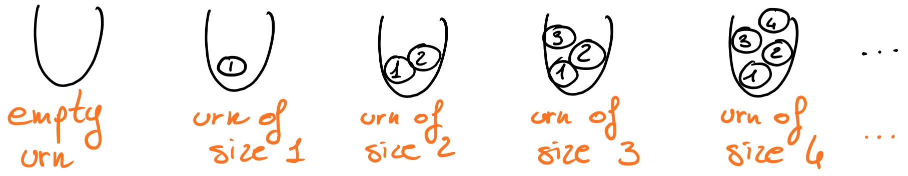
Exercise 1

$X_n :=$ size of I_1 in a uniform random set-comp. of size n.

(a) We first consider the class of urns, i.e., the collections of balls from $\{1, \dots, n\}$.
There is exactly one urn of size n for all $n \geq 0$:



Denoting with $U(z)$ the EGF for the class of urns, we have

$$U(z) = \sum_{n \geq 0} \frac{1}{n!} z^n = e^z$$

and if $U^*(z)$ is the EGF for the class of non-empty urns we have

$$U^*(z) = e^z - 1.$$

Denoting with $\mathcal{C}$ the class of set-compositions, we trivially have that

$$\mathcal{C} = \mathrm{Seq}(\underbrace{\mathcal{U}^*}_{\text{class of non-empty urns}})$$

[Recall that: if $\mathcal{C} = \mathcal{A} * \mathcal{B}$ then $\mathcal{C}_I = \biguplus_{\substack{J,K \\ I = J \uplus K}} \mathcal{A}_J \times \mathcal{B}_K$]

If $C(z)$ denotes the EGF of $\mathcal{C}$, then from the previous eq. we have

$$C(z) = \frac{1}{1 - (e^z - 1)} = \frac{1}{2 - e^z}$$

Now note that if we want to track the size of a urn in $\mathcal{U}^*$ we have

$$U(z,u) = \underbrace{uz}_{\text{size 1}} + \underbrace{u^2 \frac{z^2}{2!}}_{\text{size 2}} + \underbrace{u^3 \frac{z^3}{3!}}_{\text{size 3}} + \dots = \exp(zu) - 1$$

and so, from the formula for the product of EGF for labelled classes we obtain

$$C(z,u) = \frac{\exp(zu)-1}{2-e^z}.$$

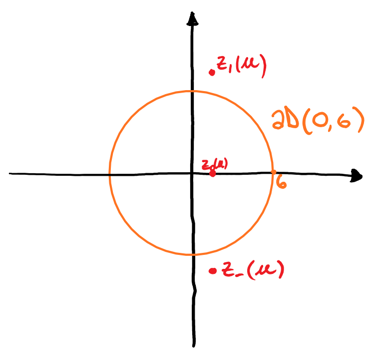
(b) The poles of $C(z,u)$ are the solutions of

$$e^z - 2 = 0$$

that are

$$z_k = \log 2 + i 2\pi k \quad , k \in \mathbb{Z}$$

Moreover, $C(z,u)$ is meromorphic in $\mathbb{C}$ since $\exp(zu)-1$ is holomorphic for every $u \in \mathbb{C}$.



Note that $z_0 \in \partial D(0,6)$ and $z_k \notin \partial D(0,6)$ $\forall k \in \mathbb{Z}^*$.

Using the residue theorem, we can conclude that

$$\frac{1}{2\pi i} \int_{\partial D(0,6)} \frac{C(z,u)}{z^{n+1}}\, dz = [z^n]\, C(z,u) + \operatorname{Res}\left(\frac{C(z,u)}{z^{n+1}}, z_0 \right)$$

(1) We first determine $\operatorname{Res}\left(\frac{C(z,u)}{z^{n+1}}, \underset{\log 2}{\overset{\shortparallel}{z_0}} \right)$:

Note that $\frac{d}{dz}\left(z^{n+1}(2-\exp(z)) \right)\Big| = \left((n+1) z^n (2-\exp(z)) + z^{n+1}(-\exp(z)) \right)\Big| = -2(\log 2)^{n+1} \neq 0$

Note that $\frac{d}{dz}\left(z^{n+1}(2-\exp(z))\right)\Big|_{z=z_0} = \left((n+1)z^n(2-\exp(z)) + z^{n+1}(-\exp(z))\right)\Big|_{z=z_0} = -2(\log 2)^{n+1} \neq 0$

$\Rightarrow$ z_k is a simple pole

[Note that $C(z,u) = \frac{f_1(z,u)}{f_2(z,u)}$ and $(z-z_0)\frac{f_1(z)}{f_2(z)} = f_1(z)\cdot\Big(\frac{z-z_0}{f_2(z)-\underbrace{f_2(z_0)}_{0}}\Big) \xrightarrow[z\to z_0]{} \frac{f_1(z_0)}{f_2'(z_0)}$]

$$\operatorname{Res}\left(\frac{C(z,u)}{z^{n+1}}, z_0\right) = \lim_{z\to z_0}(z-z_0)\frac{C(z,u)}{z^{n+1}} = \frac{\exp(u\log 2)-1}{-2(\log 2)^{n+1}}$$

(2) We now estimate the integral:

$$\left|\frac{1}{2\pi i}\int_{\partial D(0,6)} \frac{C(z,u)}{z^{n+1}}\,dz\right| \leq \frac{1}{2\pi}\cdot 12\pi \cdot \sup_{z\in\partial D(0,6)}\left|\frac{\exp(zu)-1}{z^{n+1}(2-\exp(z))}\right| = O(6^{-n})$$

$\Rightarrow$ the integral is $O(6^{-n})$ unif. for u in a suff. small neighb. of 1.

From (1) and (2) we obtain that

$$[z^n]\,C(z,u) = \frac{\exp(u\log 2)-1}{2\log(2)}(\log 2)^{-n} + O(6^{-n})$$

(c) We get that the probability generating function is, uniformly for u in a suff. small neighb. of 1 is

$$P_n(u) = \exp(u\log 2) - 1 + O\left(\left(\frac{6}{\log 2}\right)^{-n}\right)$$

$$= \sum_{k\geq 1}\frac{(\log 2)^k}{k!}u^k + O\left(\left(\frac{6}{\log 2}\right)^{-n}\right)$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}(X_n = k) = \frac{(\log 2)^k}{k!}$$

(d) $\mathbb{P}(X_n = k) = \dfrac{\#\text{ of set comp. of size } n \text{ with } |I_1| = k}{\#\text{ of set comp. of size } n} =$

$= \dfrac{\binom{n}{k} c_{n-k}}{c_n}$

# of ways to chose the elem. of I_1

# of ways to complete the set-comp starting with I_1.

From 2: $c_n = \left(\dfrac{(\log 2)^{-n-1}}{2} + O(6^{-n}) \right) n!$

Moreover $\binom{n}{k} \sim \dfrac{n^k}{k!}$

Therefore $\lim_{n\to\infty} \mathbb{P}(X_n = k) = \lim_{n\to\infty} \dfrac{\frac{n^k}{k!} \cancel{\frac{1}{2}} \cdot (\log 2)^{\cancel{-n}+k\cancel{-1}} (n-k)!}{\cancel{\frac{(\log 2)^{-n-1}}{2}} \; n!} = \dfrac{(\log 2)^k}{k!}$

Exercise 2

$G(n,p)$ with p fixed.

Let $V_n = \#$ of isol. vertex in $G(n,p) = \sum_{i \in \{1,\dots,n\}} \mathbb{1}_{\{i \text{ is isolated}\}}$

Note that

$$\mathbb{E}[V_n] = \sum_{i\in[n]} \mathbb{P}(i \text{ is isolated}) = \sum_{i=1}^{n} (1-p)^{n-1} = (1-p)^{n-1}\, n$$

Since $p \in (0,1)$ then $\mathbb{E}[V_n] \xrightarrow{n\to\infty} 0$ and so using the first moment method we conclude that

$$\mathbb{P}(V_n = 0) \xrightarrow[n\to\infty]{} 1$$

Exercise 3

$X_n = \sum^{n-1} \mathbb{1}_{\{\sigma_i < \sigma_i < \sigma_{i+1}\}}$

$$X_n = \sum_{i=2}^{n-1} \mathbb{1}_{\{\sigma_{i-1} < \sigma_i < \sigma_{i+1}\}}$$

(a) $$\mathbb{E}[X_n] = \sum_{i=2}^{n-1} \mathbb{P}(\sigma_{i-1} < \sigma_i < \sigma_{i+1}) = \frac{n-2}{6}$$

because for trivial simmetry reasons $p = \mathbb{P}(\sigma_{i-1} < \sigma_i < \sigma_{i+1}) = \frac{1}{6}$.

(b) $$\mathrm{Var}(X_n) = \sum_{i=2}^{n-1} \mathrm{Var}\left(\mathbb{1}_{\{\sigma_{i-1}<\sigma_i<\sigma_{i+1}\}}\right) + 2\sum_{2<i<j<n-1} \mathrm{Cov}\left(\mathbb{1}_{\{\sigma_{i-1}<\sigma_i<\sigma_{i+1}\}}, \mathbb{1}_{\{\sigma_{j-1}<\sigma_j<\sigma_{j+1}\}}\right)$$

$$= \mathbb{E}[X_n] - \sum_{i=2}^{n-1} p^2 + 2\sum_{2<i<j<n-1} \underbrace{\mathrm{Cov}\left(\mathbb{1}_{\{\sigma_{i-1}<\sigma_i<\sigma_{i+1}\}}, \mathbb{1}_{\{\sigma_{j-1}<\sigma_j<\sigma_{j+1}\}}\right)}_{\mathbb{E}\left[\mathbb{1}_{\{\sigma_{i-1}<\sigma_i<\sigma_{i+1};\, \sigma_{j-1}<\sigma_j<\sigma_{j+1}\}}\right] - p^2}$$

If $|i-j|>2$ then $\mathrm{Cov}\left(\mathbb{1}_{\{\sigma_{i-1}<\sigma_i<\sigma_{i+1}\}}, \mathbb{1}_{\{\sigma_{j-1}<\sigma_j<\sigma_{j+1}\}}\right) = 0$ since the r.v. are indep.

If $j=i+1$ then $\mathrm{Cov}\left(\mathbb{1}_{\{\sigma_{i-1}<\sigma_i<\sigma_{i+1}\}}, \mathbb{1}_{\{\sigma_{j-1}<\sigma_j<\sigma_{j+1}\}}\right) = \underbrace{\mathbb{E}\left[\mathbb{1}_{\{\sigma_{i-1}<\sigma_i<\sigma_{i+1}<\sigma_{i+2}\}}\right]}_{\frac{1}{24}} - p^2 = \frac{1}{72}$ (same reasoning as before)

If $j=i+2$ then $\mathrm{Cov}\left(\mathbb{1}_{\{\sigma_{i-1}<\sigma_i<\sigma_{i+1}\}}, \mathbb{1}_{\{\sigma_{j-1}<\sigma_j<\sigma_{j+1}\}}\right) = \mathbb{E}\left[\mathbb{1}_{\{\sigma_{i-1}<\sigma_i<\sigma_{i+1}<\sigma_{i+2}<\sigma_{i+3}\}}\right] - p^2 = -\frac{7}{6\cdot 5\cdot 4\cdot 3}$

$$\Rightarrow \mathrm{Var}(X_n) = \frac{n-2}{6} - \frac{n-2}{36} + 2(n-3)\frac{1}{72} - 2(n-4)\frac{7}{6\cdot5\cdot4\cdot3} = \frac{23}{180}n + O(1)$$

$$\Rightarrow v = \frac{23}{180}.$$

Consequently, since X_n is a seq. of r.v. s.t. $\lim_{n\to\infty} \frac{\mathrm{Var}(X_n)}{\mathbb{E}[X_n]^2} = 0$ then $\mathbb{P}(X_n = 0) \longrightarrow 0$ and even more, $\frac{X_n}{\mathbb{E}[X_n]} \xrightarrow[n\to\infty]{P} 1$ (corollaries of Cheb's ineq.)

(c) By Chebischev's inequality

$$\mathbb{P}\left(|X_n - \mathbb{E}[X_n]| \geq \varepsilon\right) \leq \frac{\mathrm{Var}(X_n)}{\varepsilon^2}$$

Therefore for $\varepsilon = \frac{n}{10}$ we have

$$\mathbb{P}\left(|X_n - \mathbb{E}[X_n]| \geq \frac{n}{10}\right) \leq \frac{100v}{n} + O(1/n^2)$$

(d) We sample a uniform permutation of size n using n independent random variables $U_1, \dots, U_n$ uniform in the interval $(0,1)$ [see lecture notes for this construction]

We set

$$F_i := \mathbb{E}\left[X_n \mid U_1, \dots, U_i\right] \quad , \quad \forall i \in [n].$$

Let $D_i = F_i - F_{i-1}$. Note that $\sum_{i=1}^{n} D_i = X_n - \mathbb{E}[X_n]$.

Since all the random variables $\{U_i\}_{i=1}^n$ are independent then the family $(D_i)_{1 \le i \le n}$ is a multiplicative system.

Moreover, changing any U_i changes the value of Z_n by at most 2.

Thus, we have $|D_i| \le 2$ a.s. . From Azuma's inequality we get that, for any $t_n > 0$,

$$\mathbb{P}\left(\left|X_n - \mathbb{E}[X_n]\right| \geq \frac{n}{10}\right) \leq 2 \cdot \exp\left(-\frac{n}{800}\right).$$

Exercise 4

$X_n = \sum_{i \in I_n} \mathbb{1}[A_{i,n}]$ $\qquad$ D_n collection of 2-elem subsets of I_n

$$\{i,j\} \notin D_n \Rightarrow A_{i,n} \perp\!\!\!\perp A_{j,n}$$

(a) $\quad \Delta := \sum_{\substack{i,j \in I \\ i \neq j, \{i,j\} \in D}} \mathbb{P}(A_i \cap A_j)$

0 if $i,j \notin D$ (because indep.)

$$\mathrm{Var}(X) = \sum_{i\in I_n} \mathrm{Var}(\mathbb{1}[A_i]) + 2\sum_{i<j\in I_n} \underbrace{\mathrm{Cov}(\mathbb{1}[A_i], \mathbb{1}[A_j])}_{}$$

$= 0$ if $i,j \notin D$ (because indep.)

$$= \mathbb{E}[X] - \sum_{i\in I_n} \mathbb{P}(A_i)^2 + \underbrace{\sum_{\substack{i,j\in I \\ i\neq j,\ \{i,j\}\in D}} \mathbb{P}(A_i \cap A_j) - \mathbb{P}(A_i)\mathbb{P}(A_j)}_{\Delta}$$

$$\leq \mathbb{E}[X] + \Delta$$

(b) Assume $\mathbb{E}[X] \longrightarrow \infty$ and $\Delta = o(\mathbb{E}[X]^2)$ then by Chebischev's ineq

$$\mathbb{P}\left(|X-\mathbb{E}[X]| > \varepsilon\,\mathbb{E}[X]\right) \leq \frac{1}{\varepsilon^2}\,\frac{\mathrm{Var}(X)}{\mathbb{E}[X]^2} \leq \frac{1}{\varepsilon^2}\cdot\left(\frac{1}{\mathbb{E}[X]} + \frac{\Delta}{\mathbb{E}[X]^2}\right) \longrightarrow 0.$$

(c) $\Delta^* = \Delta^*(i) := \sum_{j:\{i,j\}\in D} \mathbb{P}(A_j \mid A_i)$ does not depend on i.

Assumption: $\mathbb{E}[X] \longrightarrow +\infty$, $\Delta^* = o(\mathbb{E}[X])$

Note that

$$\mathrm{Var}(X) = \mathbb{E}[X] - \sum_{i\in I_n} \mathbb{P}(A_i)^2 + \underbrace{\sum_{\substack{i,j\in I \\ i\neq j,\ \{i,j\}\in D}} \mathbb{P}(A_i \cap A_j) - \mathbb{P}(A_i)\mathbb{P}(A_j)}_{}$$

$$= \sum_{i:\{i,j\}\in D}\ \sum_{j:\{i,j\}\in D} \mathbb{P}(A_j \mid A_i)\,\mathbb{P}(A_i)$$

$$= \sum_{i:\{i,j\}\in D} \mathbb{P}(A_i) \underbrace{\sum_{j:\{i,j\}\in D} \mathbb{P}(A_j, A_i)}_{=\Delta^*}$$

$$= \Delta^* \cdot \underbrace{\sum_{i:\{i,j\}\in D} \mathbb{E}[\mathbb{1}_{A_i}]}_{\leq \mathbb{E}[X]}$$

$$\Longrightarrow \quad \mathrm{Var}(X) \leq \mathbb{E}[X]\,(1+\Delta^*)$$

$$\Longrightarrow \quad \mathbb{P}\left(|X-\mathbb{E}[X]| > \varepsilon\,\mathbb{E}[X]\right) \leq \frac{1}{\varepsilon^2}\,\frac{\mathrm{Var}(X)}{\mathbb{E}[X]^2} \leq \frac{1}{\varepsilon^2}\cdot\left(\frac{1+\Delta^*}{\mathbb{E}[X]}\right) \longrightarrow 0.$$

(d) $A_i = \{\omega \mid (\omega_i, \omega_{i+1}, \omega_{i+2}) = (a,b,a)\}, \forall i \in [n]$

Let $D = \{(i,j) \in [n^2] \mid |i-j| \leq 2\}$ then

$$\text{if } \{i,j\} \notin D \Rightarrow A_i \perp\!\!\!\perp A_j$$

Moreover, setting $X = X_n = \sum_{i=1}^{n} \mathbb{1}_{A_i}$, we have

$$\mathbb{E}[X_n] = \sum_{i=1}^{n} \mathbb{P}(A_i) = n\left(\frac{1}{2}\right)^3 \longrightarrow +\infty$$

and

$$\Delta = \sum_{\{i,j\} \in D} \mathbb{P}(A_i \cap A_j) = \underbrace{n \cdot 0}_{\text{when } |i-j|=1} + \underbrace{n \cdot C_2}_{\text{when } |i-j|=2} = C \cdot n$$

Note that $\dfrac{\Delta}{\mathbb{E}[X]^2} \longrightarrow 0 \Rightarrow \Delta = o\left(\mathbb{E}[X]^2\right)$

We conclude using point (b).