Exercise	Points	Max
1		6
2		2
3		5
4		4
Total		**17**
Note		

Exercise 1
We recall that a set-composition of size n is an ordered sequence $(I_1, \dots, I_l)$ of disjoint non-empty sets whose union is $\{1, \dots, n\}$. We are interested in the random variable X_n, defined as the size of I_1 in a uniform random set-composition of size n

(a) (*1 point*) Let $\mathcal{C}$ be the (unlabeled or labeled?) combinatorial class of non-empty set-compositions and let $C(z, u)$ be its (ordinary or exponential?) bivariate generating series, where the exponent of u is the size of the first part of the compositition. Show that it is given by

$$C(z,u) = \frac{\exp(zu) - 1}{2 - \exp(z)}.$$

(b) (*2 points*) What are the poles of $C(z, u)$ for fixed $u \neq 0$? Show that

$$[z^n]C(z,u) = \frac{\exp(u\log(2)) - 1}{2\log(2)} (\log(2))^{-n} + O(6^{-n}).$$

(c) (*1 point*) Fix $k \geq 1$. Using probability generating functions, compute $\lim_{n\to\infty} \mathbb{P}(X_n = k)$.

(d) (*2 points*) We call c_n the number of set compositions of size n. Show by combinatorial arguments that

$$\mathbb{P}(X_n = k) = \binom{n}{k} \frac{c_{n-k}}{c_n}.$$

Use question 2. to recover the limit of $\mathbb{P}(X_n = k)$ in another way.

Exercise 2

Let p be a fixed parameter in $(0, 1)$ and consider Erdős-Rényi random graph $G(n, p)$. We call *isolated vertex* in a graph a vertex which is connected to no other.

(a) (*2 points*) Prove, that with probability tending to 1, the random graph $G(n, p)$ does not contain any isolated vertex.

Exercise 3

A *double ascent* in a permutation σ of size n is an integer between 2 and $n-1$ such that $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$. We let X_n be the number of double ascents in a uniform random permutation of size n.

(a) (*1 point*) Compute $\mathbb{E}[X_n]$.

(b) (*2 points*) Show that $\mathrm{Var}(X_n) = vn + O(1)$ for a constant v to be determined explicitly. What can you conclude?

(c) (*.5 points*) Deduce from the previous question an asymptotic upper bound for

$$\mathbb{P}\big[|X_n - \mathbb{E}[X_n]| \geq \tfrac{n}{10}\big].$$

(d) (*1.5 points*) Show that

$$\mathbb{P}\big[|X_n - \mathbb{E}[X_n]| \geq \tfrac{n}{10}\big] \leq 2\exp\big(-\tfrac{n}{800}\big).$$

Exercise 4
We use the notation $\mathbf{1}[A]$ for the indicator function of an event A. We consider a sequence of random variable X_n such that each X_n decomposes as $X_n = \sum_{i \in I_n} \mathbf{1}[A_{i,n}]$, for some collection of events $(A_{i,n})_{i \in I_n}$. For each $n \geq 0$, assume that we are given a set D_n of 2-element subsets of I_n such that

$$\{i,j\} \notin D_n \;\Rightarrow\; A_{i,n} \text{ and } A_{j,n} \text{ are independent.}$$

To simplify notation, we now drop the index n and write X, A_i, I, D instead of X_n, $A_{i,n}$, D_n, I_n, ...

(a) (*1 point*) Define

$$\Delta = \sum_{\substack{i,j \in I \\ i \neq j \text{and } \{i,j\} \in D}} \mathbb{P}(A_i \cap A_j).$$

Prove that $\mathrm{Var}(X) \leq \mathbb{E}(X) + \Delta$.

(b) (*.5 points*) We assume $\mathbb{E}(X) \to \infty$ and $\Delta = o(\mathbb{E}(X)^2)$. Show that $\frac{X}{\mathbb{E}(X)}$ tends to 1 in probability.

(c) (*1 point*) We further assume that $\Delta^* = \Delta^*(i) := \sum_{j:\{i,j\} \in D} \mathbb{P}(A_j | A_i)$ does not depend on i in I (*e.g.*, because of some symmetry). Show that, if $\mathbb{E}(X) \to \infty$ and if $\Delta^* = o(\mathbb{E}(X))$, then $\frac{X}{\mathbb{E}(X)}$ tends to 1 in probability.

(d) (*1.5 points*) We now consider a concrete model. Let $w = (w_1, \dots, w_n)$ be a uniform random word of size n with letters in the alphabet $\Omega = \{a, b\}$ and, for i in $\{1, \dots, n\}$, we let A_i be the event that $(w_i, w_{i+1}, w_{i+2}) = (a, b, a)$, where, by convention $w_{n+1} = w_1$ and $w_{n+2} = w_2$. As above, we denote $X = \sum_{i=1}^{n} A_i$.

Show that $\frac{X}{\mathbb{E}(X)}$ tends to 1 in probability.

Name: ____ __

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