Combinatorics of Words (Fall 2019)
Exercise sheet 1

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Time-line: This exercise sheet will be on-line on Thursday, October 10, 2019. You should prepare it for Thursday, October 17, 2019. The lecture of October 17 will be dedicated to the solution of this exercise sheet. There is no need to hand in your solution in advance (solutions are not graded), just bring it with you on October 17.

Presentation of solutions: There is the possibility for the students to present at the blackboard the solution of one exercise. This is completely voluntary and the solution will be not graded. It is simply an opportunity to improve the presentation of a mathematical result in front of other mathematicians.

If you are interested in presenting the solution of one exercise, please contact the assistant Jacopo Borga (jacopo.borga@math.uzh.ch) communicating to him which exercise you want to solve (for exercise 5 you can chose to present either the first four points or the last four points of the exercise). Obviously: first come, first served!

If you decide you want to present the solution to one exercise, but would be happy to have feedback on your solution before presenting it in front of the other students, you are invited to discuss your solution with Jacopo Borga, between October 11 and October 16.

Exercise 1: Prouhet's problem and the Thue-Morse word
Recall the following problem (a particular case of Prouhet's problem) presented in the introductory lecture:

For $n \geq 3$, find a bipartition $\{a_i, 1 \leq i \leq r\} \uplus \{b_i, 1 \leq i \leq r\}$ of $\{0, 1, 2, \ldots, 2^n - 1\}$ (with $r = 2^{n-1}$) such that $\begin{cases} a_1 + a_2 + \cdots + a_r = b_1 + b_2 + \cdots + b_r \\ a_1^2 + a_2^2 + \cdots + a_r^2 = b_1^2 + b_2^2 + \cdots + b_r^2 \end{cases}$.

In this exercise, we want to prove the claim of the lecture that a solution is given by choosing the a_i's (resp. b_i's) to be the numbers whose binary expansion have an even (resp. odd) number of 1.

1. For every i which is a multiple of 4, find the partition $\{a, a'\} \uplus \{b, b'\}$ of $\{i, i+1, i+2, i+3\}$ which is induced by the parity of the number of 1 in the binary expansion. Show that this partition is such that $a + a' = b + b'$.

2. For every i which is a multiple of 8, find the partition $\{a, a', a'', a'''\} \uplus \{b, b', b'', b'''\}$ of $\{i, i+1, \ldots, i+7\}$ which is induced by the parity of the number of 1 in the binary expansion. Show that this partition is such that $a^2 + a'^2 + a''^2 + a'''^2 = b^2 + b'^2 + b''^2 + b'''^2$.

3. Prove that the bipartition $\{a_i, 1 \leq i \leq r\} \uplus \{b_i, 1 \leq i \leq r\}$ of $\{0, 1, 2, \ldots, 2^n - 1\}$ defined above indeed provides a solution to the stated problem.

(Recall from the lecture that the infinite word $t = t_0 t_1 \ldots$ such that $t_i = a$ (resp. b) if the binary expansion of i contains an even (resp. odd) number of 1 is the Thue-Morse word.)

Exercise 2: Levi's lemma
Prove the following statement, sometimes called *Levi's lemma* or *equidivisibility property of the free monoid*:

> For any words u, v, u', v' of Σ^*, if $uv = u'v'$, then there exists $w \in \Sigma^*$ such that $u = u'w$ and $v' = wv$ or there exists $w \in \Sigma^*$ such that $u' = uw$ and $v = wv'$.

Exercise 3: A code
Recall that a code is the minimum generating set of a free submonoid of Σ^*. In other words, a finite set S is a code if it is minimal for inclusion among the sets generating S^*, and there is no relation among its elements (meaning there is no pair of distinct sequences $(s_1, \dots, s_i)$ and $(s'_1, \dots, s'_j)$ of words of S such that the words $s_1 \dots s_i$ and $s'_1 \dots s'_j$ are equal).
Show that the set $\{aa, ba, baa, bb, bba\}$ is a code.

Exercise 4: Isomorphisms of monoids
Let $\varphi : M \to N$ be a morphism of monoids. Assume that φ is also a bijection between M and N. Show that $\varphi^{-1} : N \to M$ is also a morphism of monoids.

Exercise 5: A characterization of free submonoids of Σ^*
The goal of this exercise is to prove the following claim from the lecture:
A submonoid M of Σ^* is free when

$$\begin{gathered}\text{for any } w \in \Sigma^*, \\ w \in M \text{ if and only if there exist } p, s \in M \text{ such that } pw \in M \text{ and } ws \in M.\end{gathered} \qquad (\star\star)$$

Denote by G the minimal generating set of M. Assume first that $(\star\star)$ holds.

1. Assume there is a relation $g_1 g_2 \dots g_n = h_1 h_2 \dots h_k$ among elements of G. Without loss of generality, assume further that $|g_1| \leq |h_1|$. Show that there is a word w such that $h_1 = g_1 w$ and $w \in M$.
2. Prove by contradiction that $w = \varepsilon$. (*Hint:* use the description of G as $(M \setminus \{\varepsilon\}) \setminus (M \setminus \{\varepsilon\})^2$.)
3. Conclude that $g_1 = h_1$.
4. Explain how the above argument can be iterated to show that the relation $g_1 g_2 \dots g_n = h_1 h_2 \dots h_k$ is trivial.

This ensures that M is free.

Assume next that M is free. This means that there exist an alphabet A and a mapping $\alpha : A \to M$ such that the unique morphism φ from A^* to M satisfying $\varphi \circ i = \alpha$ is a bijection. (As usual, i is the canonical injection from A to A^*).

5. For $w \in M$, find p and s to satisfy the right-hand-side statement in the equivalence $(\star\star)$.
6. For $w \in \Sigma^*$, assume there exist p and s such that $pw \in M$ and $ws \in M$. Define the four words of A^*

   $$x = \varphi^{-1}(p), \quad y = \varphi^{-1}(ws), \quad z = \varphi^{-1}(pw) \quad \textit{and} \quad t = \varphi^{-1}(s).$$

   Show that $xy = zt$.
7. Show that x is a prefix of z. (*Hint:* if z were a proper prefix of x, you would reach a contradiction.)
8. This allows to write $z = xu$, for some $u \in A^*$. Check that $\varphi(u) \in M$, and prove that $\varphi(u) = w$.

This ensures that $(\star\star)$ holds.