Combinatorics of Words (Fall 2019): Homework

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Time-line and comments: This homework will be on-line on Friday, November 8, 2019. You must hand in your solution to Jacopo Borga at the latest on Monday, November 25, 2019.

You can either bring your solution to Jacopo's office (Y27G25), or drop it in his mailbox (in Y27, G floor), or send it by e-mail to `jacopo.borga@math.uzh.ch` (either pdf file, or *clean* scan of *clean* handwritten solution).

These exercises will be discussed during the lecture of December 5, 2019. You can also have your solution back on December 5 (or any lecture after).

You can prepare the solution to the homework on your own, or in pairs, but not in larger groups.

This homework will be graded. If H is your grade for the homework and F your grade at the final exam, your grade for this module will be $\max(F, \frac{2F+H}{3})$. In particular, your grade for the homework can *only improve* your grade for the module.

Exercise 1: Border of words
We define that a word u is a *border* of a word v if u is both a proper prefix and a suffix of v. (Recall that *proper* means here different from v.) In addition, we say that a word is *unbordered* if its only border is the empty word.

1. Justify that if u is a border of v, then u is a proper suffix of v.
2. Let w be a word having a non-empty border. Justify that we can consider the word u which is the shortest non-empty border of w.
3. Show that u is unbordered.
4. Deduce that there exists a word w' (possibly empty) such that $w = uw'u$.
5. Is the decomposition of w as $uw'u$ possible for *every* border u of w? If yes, provide a proof. If no, provide a counter-example.

Exercise 2: An equation with constants
The goal of this exercise is to solve the equation $XaXbY = aXYbX$, on any alphabet Σ containing $\{a, b\}$ and on the set of unknowns $\{X, Y\}$. More precisely, we will prove that its solutions are the morphisms of monoids $\alpha : (\Sigma \cup \{X, Y\})^\star \to \Sigma^\star$ fixing a and b, and such that $\alpha(X) = a^i$ and $\alpha(Y) = a^i(ba^i)^j$ for some $i, j \geq 0$.

1. Check first that any α of the form described above is indeed a solution of $XaXbY = aXYbX$.

   For the next two questions, consider a solution α of $XaXbY = aXYbX$.

2. By considering a prefix of $\alpha(XaXbY) = \alpha(aXYbX)$ of well-chosen length, show that $\alpha(X) = a^i$ for some $i \geq 0$.
3. By considering a suffix of $\alpha(XaXbY) = \alpha(aXYbX)$, show that α (restricted to the domain $(\Sigma \cup \{Y\})^\star$) is also a solution of $a^ibY = Yba^i$.

   In the next two questions, we solve the equation $a^ibY = Yba^i$.

4. Check first that if $\alpha(Y) = (a^ib)^ja^i$ for some $j \geq 0$, then α is a solution of $a^ibY = Yba^i$.
5. Show that any solution of $a^ibY = Yba^i$ is of this form.
   Hint: A proof can be done by induction on the number of letters b in $\alpha(Y)$.
6. Conclude on the resolution of $XaXbY = aXYbX$.

Exercise 3: Finite Fibonacci words

The sequence $(f_n)_{n\geq 0}$ of *finite Fibonacci words* is defined as follows:

$$f_0 = a, \qquad f_1 = ab, \qquad f_n = f_{n-1} \cdot f_{n-2} \text{ for all } n \geq 2.$$

We also recall[1] that the sequence $(F_n)_{n\geq 0}$ of Fibonacci numbers can be defined by

$$F_{-2} = 0, \qquad F_{-1} = 1, \qquad F_n = F_{n-1} + F_{n-2} \text{ for all } n \geq 0.$$

1. Show that for all $n \geq 0$, $|f_n| = F_n$. (Recall that $|f_n|$ is the length of f_n.)
2. Show that for all $n \geq 0$, $|f_n|_a = F_{n-1}$ and $|f_n|_b = F_{n-2}$. (Recall that $|f_n|_x$ denotes the number of letters x in f_n.)

   The *Fibonacci morphism* φ (from $\{a,b\}^\star$ to $\{a,b\}^\star$) is defined by $\varphi(a) = ab$ and $\varphi(b) = a$. It enjoys many nice properties.
3. Show that for any $u \in \{a,b\}^+$, $\varphi(u)$ starts with the letter a.
4. Using the previous question, show that φ is injective.
5. Recall that a word is *primitive* if it is not the power of a smaller word. Show that for any word $w \in \{a,b\}^\star$, w is primitive if and only if $\varphi(w)$ is primitive.
6. The *mirror image* of a word $w = w_1 w_2 \dots w_n$ is the word $\overleftarrow{w} = w_n \dots w_2 w_1$. A word w is a *palindrome* if it is equal to its mirror image, that is to say $w = \overleftarrow{w}$. Show that for any word $w \in \{a,b\}^\star$, w is a palindrome if and only if $\varphi(w)a$ is a palindrome.

   The Fibonacci morphism can be used to prove some properties of Fibonacci words and of their "long prefixes".
7. Prove by induction that for all $n \geq 0$, $f_n = \varphi^n(a)$. (Recall the usual convention that φ^0 is the identity map.)
8. Deduce that for all $n \geq 0$, f_n is a primitive word.
9. For any $n \geq 1$, let g_n be the prefix of length $F_n - 2$ of f_n. Show that, if $n \geq 1$ is ~~even (resp. odd)~~ **odd (resp. even)**, then $f_n = g_n ab$ (resp. $f_n = g_n ba$).
10. Prove that the sequence of words $(g_n)_{n\geq 1}$ is characterized by $g_1 = \varepsilon$ and for all $n \geq 1$, $g_{n+1} = \varphi(g_n)a$.
11. From the previous questions, it follows that the words g_n enjoy a nice property. Which one? (and why?)

Exercise 4: Periods and examples

All periods in this exercise are intended as periods in the strong sense (*i.e.* as *the* period).

1. For any $p \geq 2$, find two words u and v on alphabet $\{a,b\}$, both of period p, such that uv is not of period p.
2. Is it also possible to find two such words for $p = 1$? Either give an example, or provide a proof of non-existence.
3. Find a sequence of words $(u_n)_{n\geq 2}$ on alphabet $\{a,b\}$, such that, for all n, $|u_n| = n$ and the period of u_n is 1.
4. Find a sequence of words $(v_n)_{n\geq 1}$ on alphabet $\{a,b\}$, such that, for all n, $|v_n| = n$ and v_n has no period.

[1] with a small twist on the initial conditions which is explained by the first question of this exercise...