Limits of random permutations

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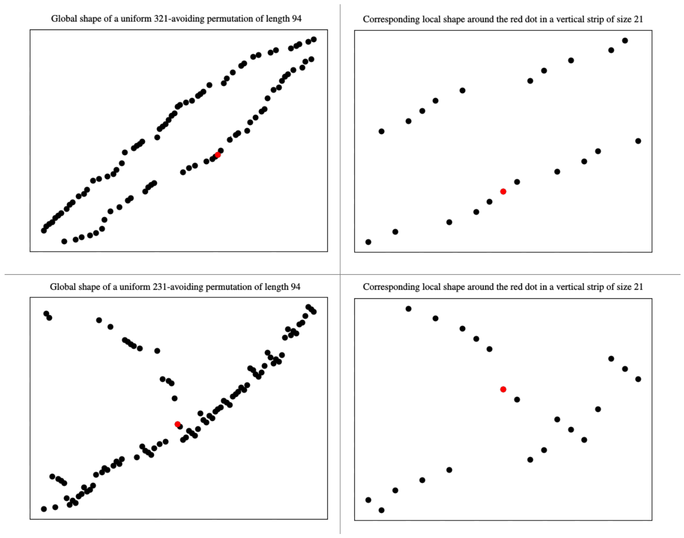


Figure 1: Scaling and local limits for two models of random permutations.

The study of random combinatorial structures (like permutations, graphs, partitions, laminations, tessellations,...) is one of the most prolific topics at the interface of combinatorics and probability theory. This research project deals with the study of random permutations, which are random total orders of the numbers from 1 to n. Owing to their simplicity, permutations are very natural models for problems arising in applied fields such as computer science, biology, and physics. We mainly focus on the study of the limiting behavior of random permutations, but we also investigate random walks and random planar maps.

This research proposal is structured as follows:

- in Section 1 we sum up the current state of the research concerning limits of discrete structures and the study of permutation patterns;
- in Section 2 we briefly explain the new notion of local convergence introduced by the applicant during the first year of Ph.D in [16];
- in Section 3 we list the projects in which the applicant was/is currently involved;
- in Section 4 we focus on the objectives for future projects and we discuss the relevance and significance of the whole project;
- in Section 5 we conclude with a summarizing schema of the project.

1 State of the art

1.1 Limits of discrete structures

For large combinatorial structures, two main notions of convergence can be defined: scaling limits and local limits. Informally, in the scaling limits approach we look at the objects from a global point of view (after a rescaling of the distances between points of the objects), while the study of local limits provides knowledge about the objects in a neighborhood around a marked point (without rescaling distances).

In Figure 1 we can see the global (on the left-hand side) and the local (on the right-hand side) behavior for two random permutations avoiding the pattern 321 and 231 respectively (see Section 1.2 for a precise definition of pattern avoiding permutations). For scaling limits, we consider the diagram[1] of the whole permutation and we rescale the distances between points by a factor $1/n$. For local limits, we consider a vertical strip of fixed size around the red point (chosen uniformly at random) and we report on the right the corresponding induced consecutive pattern (in this case without rescaling distances). We will describe this construction more precisely later on.

In the last thirty years such local and scaling limits have been extensively studied for several discrete structures. We mention some results on trees and planar maps, for which these limits have been deeply investigated, without aiming at giving a complete list.

The systematic study of scaling limits of combinatorial structures was initiated by Aldous with the pioneering series of articles about the well-known *Continuum random tree* ([2, 3, 4]). After that, many new results have been proven, in particular about the Brownian map, which is the scaling limit of several different models of maps (see for instance [42, 45, 47]). Local limit results around the root of random trees were first implicitly proved by Otter [49], then explicitly by Kesten [40] and Aldous and Pitman [5]. Janson [38] gave a unified treatment of the local limit of simply generated random trees as the number of vertices tends to infinity. Recently, Stufler [54] also studied the local limits for large Galton–Watson trees around a uniformly chosen vertex, building on previous results of Aldous [1] and Holmgren and Janson [33]. Although implicit in many earlier works, the notion of local convergence around a random vertex (called *weak local convergence*) for random graphs has been formally introduced by Benjamini and Schramm [14] and Aldous and Steele [6].

More closely related to this research project, a notion of scaling limits for permutations, called *permutons*, has been recently introduced in [34]. They are probability measures on the unit square with uniform marginals, and they represent the scaling limit of the diagram of permutations as the size grows to infinity. This new notion of convergence has been studied (sometimes phrased in other terms) in several works (see Section 1.3) and it is deeply related to *permutation patterns* that we introduce in the next section.

1.2 Permutation patterns

Given two permutations σ and π of size n and k respectively, for some $k \leq n$, we say that σ contains π as a *pattern* if σ has a subsequence of entries $\sigma_{i_1}, \ldots, \sigma_{i_k}$ which is order-isomorphic to π. In addition, we say that σ contains π as a *consecutive pattern* if σ has a subsequence of adjacent entries order-isomorphic to π. For example, the permutation $\sigma = 1532467$ contains 1423 as a pattern (in the subsequence 1534) but not as a consecutive pattern and it contains 321 as consecutive pattern (in the subsequence 532). Finally, we say that σ *avoids* π if σ does not contain π as a pattern. We point out that the definition of a π-avoiding permutation refers to patterns and not to consecutive patterns.

[1] We write permutations of size n in one-line notation as $\sigma = \sigma_1 \ldots \sigma_n$, *i.e.*, $\sigma_i = \sigma(i)$. We will often view a permutation σ as a diagram, *i.e.*, the set of points of the Cartesian plane with coordinates (j, σ_j).

As we mentioned in the previous section, scaling limits not only give information on the global behavior of a large permutation but are also strongly related to the study of patterns in permutations: specifically, for a sequence of uniform permutations in a fixed family, it is possible to deduce the convergence of the proportion of pattern occurrences of every fixed pattern of arbitrary size from the permuton convergence, and vice-versa. Similarly, we will see in Section 2 that local limits (introduced by the applicant in [16]) are strongly related to consecutive patterns.

The study of permutation patterns was started in the '60s by Knuth [41] in relation to stack-sorting problems in computer science, and since then has become a deep area of enumerative, algebraic and analytic combinatorics (see for example [15]). More recently a new probabilistic approach to the study of permutation patterns has become popular. For instance, the general problem of studying the limiting distribution of the number of occurrences (after a suitable rescaling) of a fixed pattern π in a uniform random permutation belonging to a fixed class, when the size tends to infinity, was considered in several works of Janson [35, 36, 37]. He studied this problem in the model of uniform permutations avoiding a fixed family of patterns of size three. Other probabilistic approaches to the study of permutation patterns can be founded in the works of Bassino, Bouvel, Féray, Gerin, Maazoun, Pierrot [9, 10, 11] on substitution-closed (and other) classes, in the the work of Crane, DeSalvo and Elizalde [22] on Mallows permutations, Féray [27] on Ewens permutations and in many others articles.

Also the specific study of *consecutive* patterns is a very active field both from a combinatorial and a probabilistic point of view, with applications to computer science, biology, and physics. Classical questions like the number of descents, runs and peaks contained in a permutation can be viewed as particular examples of consecutive patterns in permutations, but more general approaches have been developed. For a complete overview, we refer to the survey on the topic by Elizalde [26].

1.3 Scaling limits for random permutations

We now mention some works related to the study of scaling limits for random permutations.

- The study of the scaling limits of a uniform random permutation avoiding a pattern of length three was initiated by Miner and Pak [48] and Madras and Pehlivan [44]. Later, with a series of two articles, Hoffman, Rizzolo and Slivken ([30, 31]) strengthened these earlier results exploring the connection of these uniform pattern-avoiding permutations to Brownian excursions. All these articles do not use the "permuton language".

- The first concrete and explicit example of convergence in the "permuton language" was by Kenyon, Kral, Radin and Winkler [39]. They studied scaling limits of random permutations in which a finite number of pattern densities has been fixed.

- Bassino, Bouvel, Féray, Gerin and Pierrot [10] showed that a sequence of uniform random separable permutations of size n converges to the *Brownian separable permuton*. In this work they introduced this new limiting object that has been later investigated by Maazoun [43].

- In a second work, Bassino, Bouvel, Féray, Gerin, Maazoun and Pierrot [11] showed that the Brownian permuton has a universal property: they consider uniform random permutations in proper substitution-closed classes and study their limiting behavior in the sense of permutons, showing that the limit is an elementary one-parameter deformation of the Brownian separable permuton. They also characterized permuton limits in terms of convergence of frequencies of pattern occurrences (see [11, Theorem 2.5]).

- With a statistical mechanical approach , Starr [53] investigated the permuton limit of Mallows permutations. This is a non-uniform model where the probability of every permutation is proportional to $q^{\text{number of inversions}}$. Treating the question as a mean-field problem, he calculated the distribution of the permuton limit (after a suitable rescaling of the parameter q).
- Rahman, Virag and Vizer [52] generalized the study of scaling limits for permutations to permutation-valued processes, *i.e.* sequences of sequences of permutations.

To the best of our knowledge, a local limit approach has not been investigated in the framework of permutations before the work of the applicant [16]. We want only to mention that Hoffman, Rizzolo and Slivken [32] used local limits results for random trees in the study of some specific properties of permutations, like the number of fixed points (*i.e.*, indexes i such that $\sigma_i = i$) of pattern-avoiding permutations. However, they didn't study general local limits results for permutations.

2 A new notion of local convergence for random permutations

During the first year of the Ph.D. I set up in [16] a new notion of local convergence for permutations in order to fill the previously described gap and showed several aspects of the local convergence for permutations. More precisely,

- I introduced a notion of local topology for permutations.

  I set up a bijection between rooted permutations, *i.e.*, permutations with a distinguished point, and total orders on finite intervals containing 0. Roughly speaking (see Fig. 2 for an example), the elements of the finite interval should be thought of as the column indices of the diagram of the permutation, shifted so that the root is in column 0. The order on that interval then corresponds to the bottom-to-top order on the points in the corresponding columns. Then, I defined the limiting objects (called *infinite rooted permutations*) as total orders on *infinite* intervals containing 0.

  In order to define a local topology, I first introduced a notion of neighborhoods around a distinguished vertex of the diagram of a permutation: these are the consecutive patterns induced by "vertical strips" around the distinguished vertex of the permutation (see Fig. 2).

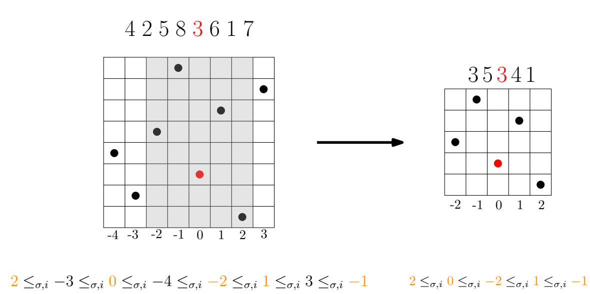


Figure 2: Neighborhood around a distinguished vertex of the diagram of a permutation with the corresponding associated total order.

  A sequence of permutations is defined to be *Benjamini–Schramm convergent* if choosing the distinguished vertex uniformly at random (for every permutation in the sequence), for every possible size of the vertical strip, the corresponding consecutive patterns are eventually equal.

- I proved a characterization of this notion of local convergence in terms of proportions of *consecutive* pattern occurrences. For the sake of comparison, recall that permuton limits have been characterized in terms of convergence of frequencies of (non-consecutive) pattern occurrences.

More precisely, denoting, for any pattern π of size k and any permutation σ of size n, the proportion of consecutive occurrences of π in σ as

$$\widetilde{c\text{-}occ}(\pi,\sigma) = \frac{\text{number of consecutive occurrences of } \pi \text{ in } \sigma}{n},$$

I proved the following:

Theorem 1. *For any $n \in \mathbb{N}$, let σ^n be a permutation of size n. Then the Benjamini–Schramm convergence for the sequence $(\sigma^n)_{n\in\mathbb{N}}$ is equivalent to the existence of an infinite vector of non-negative real numbers $(\Delta_\pi)_{\pi\in\mathcal{S}}$ such that, for all patterns $\pi \in \mathcal{S}$,*

$$\widetilde{c\text{-}occ}(\pi,\sigma^n) \to \Delta_\pi.$$

I also extended this result to the case when $(\sigma^n)_{n\in\mathbb{N}}$ is a *random* sequence.

- I characterized random limiting objects for this new topology, introducing a “shift-invariant" property reminiscent of the notion of unimodularity for random graphs.
- I exhibited some concrete applications of the developed theory in the setting of pattern-avoiding permutations.

  In particular, denoting with $Av(\rho)$ the set of ρ-avoiding permutations and with $Av^m(\rho)$ the set of ρ-avoiding permutations of size m, I proved the following:

  Theorem 2. *Let ρ be a pattern of size 3 and, for any $n \in \mathbb{N}$, let $\boldsymbol{\sigma}^n$ be a uniform random ρ-avoiding permutation*[2]*. Then we have the following convergence in probability,*

  $$\widetilde{c\text{-}occ}(\pi,\boldsymbol{\sigma}^n) \xrightarrow{P} P_\rho(\pi), \quad \textit{for all} \quad \pi \in \mathrm{Av}(\rho), \tag{2.1}$$

  where, for all $m \in \mathbb{N}$, $\big(P_\rho(\pi)\big)_{\pi\in\mathrm{Av}^m(\rho)}$ is a probability distribution on $\mathrm{Av}^m(\rho)$ described explicitly in [16].

  This theorem implies convergence in distribution (with respect to the local topology) for the random sequence $\boldsymbol{\sigma}^n$.

We mention that recently Pinsky [50, 51] studied limits of random permutations avoiding patterns of size three considering a different topology. His topology encodes the local limit of the permutation diagram around a *corner*. The two topologies are not comparable.

3 Further obtained results and works in progress

In four projects (some completed and some ongoing), we are investigating the local and scaling limits for several families of permutations.

Project A In collaboration with M. Bouvel, V. Féray and B. Stufler [20], the applicant explored a decorated tree approach for the study of (scaling and local) limits of random permutations in substitution-closed classes. Note that, as opposed to Theorem 2, we considered here **an infinite family of models**. Using a novel bijective encoding that represents permutations as forests of enriched trees, we proved local convergence of uniform random permutations in substitution-closed classes satisfying a criticality constraint. We also proved and strengthen permuton limits for these classes in a different way (we recall that the result was already proved in [11]). This project is concluded but is relevant for future projects (see in particular Project F below).

[2] Here and throughout the proposal we denote random quantities using **bold** characters.

Project B In this second project the applicant is developing a general technique to establish central limit theorems for consecutive patterns in random permutations in families having a well-behaved generating tree. We highlight that such a CLT implies local convergence results through the characterization explained in Theorem 1.

We recall that a generating tree for a family of permutations $\mathcal{C}$ is an infinite rooted tree whose vertices are the permutations of $\mathcal{C}$ (each appearing exactly once in the tree) and such that permutations of size n are at level n (the level of a vertex being the distance from the root plus one). The children of some permutation $\sigma \in \mathcal{C}$ are obtained by a specific construction from σ (for example adding a new top row with a point to the diagram of σ). Moreover, assume that for some set $\mathcal{L}$ there exists a $\mathcal{L}$-valued statistics on the permutations of $\mathcal{C}$, whose values determine the number of children in the generating tree and the values of the statistics for the children. Then we give labels to the objects of $\mathcal{C}$ which indicate the value of that statistics. An example of a generating tree for the class of 321-avoiding permutations is given in Fig. 3.

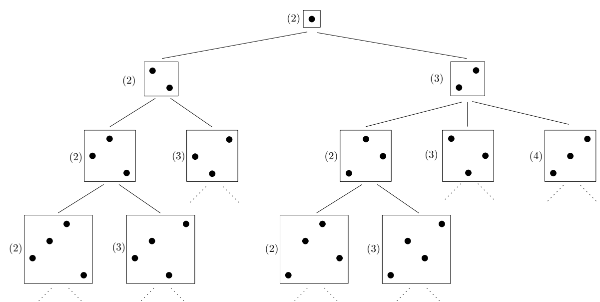


Figure 3: The generating tree for 321-avoiding permutations. The children of a permutation are obtained by adding a new point at the top of the diagram in such a way that it does not create a pattern 321. We draw only the first three levels of the tree in their entirety; for the fourth level, we only draw the children of the permutations 231 and 132. On the left-hand side of each diagram we report the corresponding label given by the statistics that counts the number of possible insertions of a new point. Note that every permutation with label (k) has k children with labels $(2), (3), \ldots, (k+1)$.

Generating trees are widely used in enumerative combinatorics (see for instance [7, 8, 13, 55]) to compute the enumeration sequences of several families of discrete objects through generating functions. In our project we will use generating trees with a completely different purpose, which is to extract some probabilistic information on the structure of the encoded permutations.

More precisely, we propose an algorithm to sample uniform permutations in such families using Markov chains on the set of labels and we describe the consecutive patterns of permutations studying local properties of the corresponding random walks. At the moment the method applies to families of permutations with a generating tree having labels in $\mathbb{Z}$; we would like to generalize it to multi-dimensional labels (i.e. labels in $\mathbb{Z}^d$). We believe that using some known results on multi-dimensional random walks conditioned to stay inside a cone (see [24]) we could prove the desired results. This last part of the project is a work in progress that can be summarized in the following question.

> Is it possible to determine a CLT for consecutive patterns (and so local limit results) in permutations encoded by multi-dimensional labels generating trees?

Project C Together with M. Maazoun, a PhD student in mathematics in the probability research group at the ENS in Lyon, we are investigating the scaling limit of Baxter permutations, one of the most studied families of pattern-avoiding permutations in the literature. The first part of this project is almost concluded.

Baxter [12] introduced these permutations while studying the fixed points of commuting continuous functions. Since then, they have been studied from an enumerative point of view (see, for instance, [21]) and from an algebraic point of view (see, for instance, [28]). Moreover, Dokos and Pak [25] explored the average limit shape of doubly alternating Baxter permutations (a smaller family with a simpler structure) and they suggested that "it would be interesting to compute the limit shape of random Baxter permutations".

Baxter permutations are well-known to be in bijections with both plane bipolar orientations and a specific family of walks conditioned to stay in the non-negative quadrant. We introduce a further new family of discrete objects (fundamental for our results), called *coalescent-walk processes*, and we determine a new bijections with the other previously mentioned families.

We are able to prove joint Benjamini–Schramm convergence for uniform objects in the four families. Furthermore, we explicitly construct **a new fractal random measure** of the unit square, called the *coalescent Baxter permuton* and we show that it is the scaling limit (in the permuton sense) of uniform Baxter permutations (for some simulations see Fig. 4).

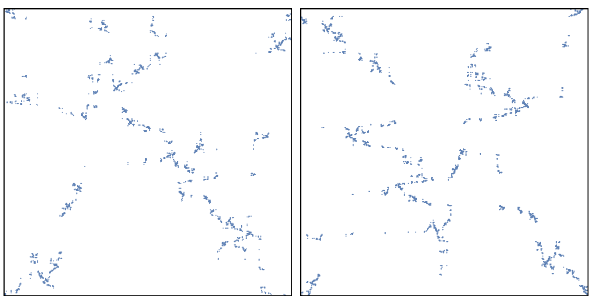


Figure 4: The diagrams of two uniform Baxter permutations of size 3253 (left) and 4520 (right).

To prove the latter result, we studied the scaling limit of the associated random coalescent-walk processes. We showed that they converge in law to a *continuous random coalescent-walk process* encoded by a perturbed version of the Tanaka stochastic differential equation. This result is **a novel connection that relates limiting objects of discrete structures to stochastic differential equations**. This result is also interesting for its connections with the work of Gwynne, Holden, Sun [29] on scaling limits (in the Peanosphere topology) of plane bipolar triangulations. We want to explore such connections in future projects.

> What is the interplay between the scaling limit for plane bipolar orientations established by Gwynne, Holden, Sun [29] (i.e. a $\sqrt{4/3}$-Liouville Quantum Gravity (LQG) surface decorated by an independent SLE_{12}) and the one introduced in our work (i.e. a continuous random coalescent-walk process encoded by a perturbed version of the Tanaka stochastic differential equation)?
> Is there a general result that relates LQG and stochastic differential equation?

Project D In collaboration with Slivken and Duchi we have been and still are investigating local and scaling limits results for (almost) square permutations.

A record in a permutation is an entry which is either larger or smaller than the entries either before or after it (there are four types of records). Entries which are not records are called internal points.

We are exploring permuton limits of uniform permutations in the classes $Sq(n,k)$ of (almost) square permutations of size $n+k$ with exactly k internal points.

We first investigated the case when $k = 0$ (see [19]), this is the class of square permutations, i.e. permutations where every point is a record. The starting point for our results is a sampling procedure for asymptotically uniform square permutations. Building on that, we characterized the global behavior by showing that square permutations have a permuton limit which can be described by a random rectangle. We also explored the fluctuations of this random rectangle, which we could describe through coupled Brownian motions. We in addition determined local limit results.

We then characterized the permuton limit of almost square permutations $Sq(n,k)$ with $k > 0$ internal points, both when k is fixed and when k tends to infinity along a negligible sequence with respect to the size of the permutation (see [17]). Here the limit is again a random rectangle but of a different nature: we showed that **a phase transition on the shape of the limiting rectangles arises** for different values of k.

We want finally to investigate in a future work the following.

> What is the scaling limit of uniform permutations in $Sq(n,k)$ when k grows fast as/faster than n?

We expect a further phase transition in the permuton limits that will end with the Lebesgue measure on the unit square.

We also mention that the applicant is currently involved in another project with R. Penaguiao (a member of the Prof. V. Féray's research group). The goal of this project is to study the feasible region of consecutive patterns of permutations and pattern-avoiding permutations (see [18] for some first results on this project).

4 Future projects

In the next year, we plan to continue the projects above but also to focus on some new aspects listed below.

Project E Local limits of permutations sampled according to an exponentially biased distribution.

Project F A new notion of semi-local convergence that interpolates between the global and the local ones.

Project G Establish permuton limits for random permutations in families having a well-behaved generating tree using the sampling procedure already introduced in Project B.

We now explain, for each of the listed projects, the motivations behind every choice, the methods/techniques we plan to use in order to reach the prescribed goal and the practical organization.

Project E Uniform random permutations have received a lot of attention in the past years and they are nowadays well-understood. However, much less is known for **non-uniform distributions**. The study of these models is a quite recent and prolific area of probabilistic combinatorics sharing connections with statistical mechanics (see [53]). In Projects A,B,C we focus on models giving positive probability only to permutations that avoid prescribed sub-structures (like classical patterns or generalized patterns), the total mass being distributed uniformly on these pattern-avoiding permutations. In this project we will focus on **exponentially biased distributions**, which are classical models considered in statistical mechanics (where often this exponentially biased distributions are called Gibbs distributions) but also in the study of planar maps (for instance, the q-Boltzmann planar map, see [46]). More precisely, we would like to consider random permutations sampled according to the following law: for a fixed family of patterns $\rho_1, \dots, \rho_m$; the probability that a random permutation σ is equal to π is

$$\mathbb{P}(\sigma = \pi) \propto q^{\sum_{i=1}^{m} \widetilde{c\text{-}occ}(\rho_i,\pi)},$$

i.e., the exponential bias is given by the proportion of consecutive occurrences of some fixed patterns.

Inspired by the (already cited) work of Kenyon, Kral, Radin and Winkler [39], we would like to develop a theory for the study of local limits of such permutations. In [39] the authors studied scaling limits of random permutations in which a finite number of (classical) pattern densities have been fixed and they elaborate a variational principle to determine the convergence. We think it is possible to study an analogous variational principle in the case of consecutive pattern densities and local limits. We would like to answer the following questions.

- What are the local limits of exponentially biased permutations?
- Is it possible to write and solve a variational principle for this problem?

Project F Local and global convergence for discrete structures are arguably the best-known notions of convergence, but at the same time other notions of convergence have been considered. For instance, the notion of semi-local convergence is used in graph theory and interpolates between the local and the global one. An example is the recent work of Curien and Le Gall [23], where they studied a notion of semi-local convergence for uniform triangulations, obtaining a new limiting object called *Brownian plane*. They also showed some interesting relations with the well-known Brownian map (the scaling limit) and the Uniform infinite planar triangulation (the local limit). We think it is possible to introduce a notion of **semi-local convergence** also for random permutations that will be useful to investigate the relation between permutons (scaling limit) and infinite rooted permutations (local limits). Moreover, as global convergence is in connection with permutation patterns and local convergence is in connection with consecutive permutation patterns, semi-local convergence would be characterized in terms of convergence of **semi-consecutive patterns**, *i.e.,* patterns determined by a set of indexes with some restrictions on the maximal distance between them. Therefore, the main questions are

- How can a notion of semi-local convergence for permutations be defined?
- Can it be characterized in terms of semi-consecutive patterns?
- What is the relationship between the different types of convergence and the corresponding limiting objects?

Finally, we believe that we could apply some semi-local results on trees obtained in the paper with Bouvel, Féray and Stufler (see Project A) in order to prove the semi-local convergence for families of permutations.

Project G In Project C we discussed a technique to sample uniform permutations in families encoded by generating trees using Markov chains on the set of labels. We believe that this sampling procedure is useful not only to establish CLTs but also to investigate permuton limits. A first evidence of this belief is the work in Project C. Indeed Baxter permutations have a well-behaved two-dimensional labeled generating tree. We did not use this encoding through generating trees in Project C, since in this specific case nicer bijection are available; but we think that also the generating tree approach would work. We therefore want to investigate the following.

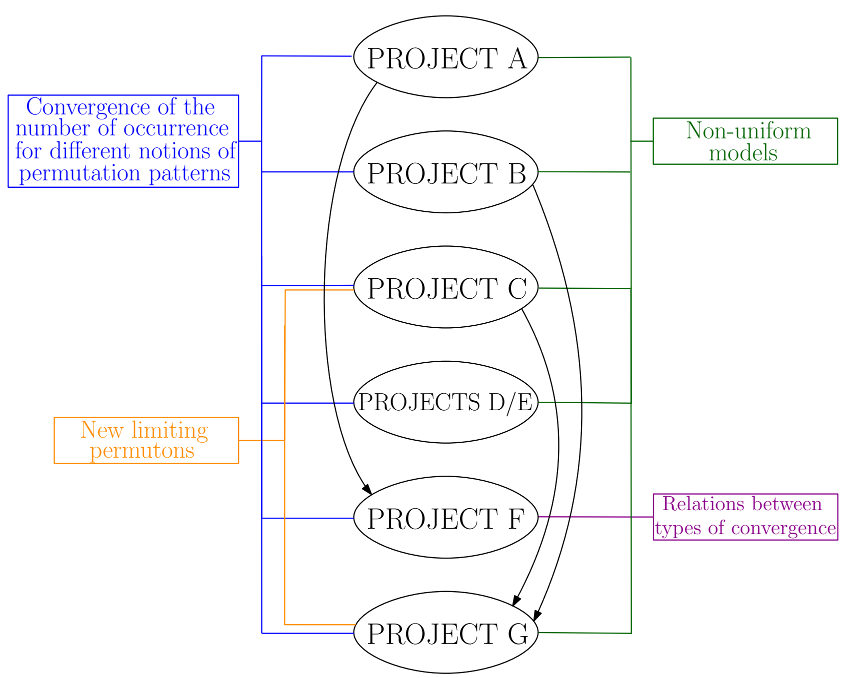
> Can we establish permuton limits for random permutations in families having a well-behaved generating tree?

5 A summarizing schema

This research project has four main relevant objectives.

- The study of the convergence of the number of occurrences for different notions of permutation patterns (classical permutation patterns, consecutive permutation patterns, semi-local permutation patterns).
- The discovery of new limiting permutons.
- The study of non-uniform models of random permutations.
- The investigation of the relations between the different notions of convergence for permutations.

These objectives will be reached as an integration of the different projects cited before. In the schema below we sum up the various relationships among the projects and our objectives.



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