Exercise	Points	Max
1		2
2		3
3		4.5
4		3
5		10.5
Total		**23**

Authorized documents:
All your personal notes about this lecture and associated exercises.
Also, all documents posted on the websites of the lecture.
No books nor other lecture notes however.

Exercises are independent, and may be solved in the order you prefer.

Periods and factors

Exercise 1
Let w be a finite word, and p be a period of w.

Show that for any factor u of w of length at least $p+1$, it holds that p is a period of u. *(2 points)*

Period and primitivity

Exercise 2
Let w be a finite word, and $p \geq 1$ be the period of w (in other words, the *smallest* period of w), assuming it exists. Let u be the prefix of w of length p.

Show that u is a primitive word. (Hint: you can do a proof by contradiction.) *(3 points)*

Lexicographic order

Exercise 3
Let $<$ denote the strict lexicographic order on Σ^* (for Σ a totally ordered alphabet). The goal of this exercise is to prove that, for all $u, v, w \in \Sigma^*$,

$$u < w < uv \text{ if and only if there exists a non-empty word } x \text{ such that } x < v \text{ and } w = ux.$$

Hint: A property of the lexicographic order (seen in the lecture) can be useful in this exercise; namely, that for all $u, v, w \in \Sigma^*$, $u < v$ if and only if $wu < wv$.
(You **do not have to prove** this property in your solution of the exam, but you **can use** it.)

(a) (*1 point*) Show first the implication from right to left.

(b) (*2 points*) To prove the implication from left to right, first show by contradiction that, under the assumptions $u < w < uv$, u is a prefix of w.

This allows to write $w = ux$ for some $x \in \Sigma^*$.

(c) (*0.5 points*) Justify that x is non-empty.

(d) (*1 point*) Show that $x < v$.

Palindromic property of prefixes of the Thue-Morse word

Exercise 4

Let (u_n) and (v_n) be the sequences of finite words (already presented in the lecture) defined by $u_0 = a$, $v_0 = b$, and for all $n \geq 0$, $u_{n+1} = u_n v_n$, $v_{n+1} = v_n u_n$.

Prove that for all n, u_{2n} and v_{2n} are palindromes and $\overleftarrow{u_{2n+1}} = v_{2n+1}$. *(3 points)*

Sturmian words are balanced

Exercise 5

The purpose of this exercise is to show that every Sturmian word is balanced.

We proceed by contradiction. Let w be a Sturmian word (on the alphabet $\{a, b\}$), and assume that w is unbalanced. We have seen in the lecture and the second exercise sheet that there must then exist a palindrome v such that ava and bvb are both factors of w. (You **do not have to prove** this property in your solution of the exam, but you **can use** it.)

We say that a finite factor u of w has one (resp. two) continuations when only one of the factors ua and ub occurs in w (resp. when both ua and ub occur in w).

(a) (*1 point*) Recalling that w is Sturmian, show that, for any n, among all factors of length n of w, one has two continuations and all others have one continuation.

(b) (*0.5 points*) Justify that v is the only factor of w of length $|v|$ having two continuations.

(c) (*0.5 points*) Deduce that the only factor of w of length $n = |v| + 1$ having two continuations is either av or bv.

W.l.o.g., we assume from now on that av has two continuations.

(d) (*1 point*) For each of the words ava, avb, bva and bvb, determine whether it is a factor of w or not.

In the following questions, we consider a factor u of length $2n$ of w which starts with bv.

(e) (*1 point*) Justify that $u = bvbv'$ for some word v' of length $|v|$.

(f) (*2 points*) Prove by contradiction that av does not occur as a factor in u. (Hint: First argue that the occurrence of av in $bvbv'$ splits $bvbv'$ as $bpaxbys$ with $v = pax$, $v = xby$ and $v' = ys$. Then, recalling that v is a palindrome, show that the letter at position $|x| + 1$ in v is both an a and a b.)

(g) (*1 point*) For any i with $1 \leq i \leq n + 1$, let $u^{(i)} = u_i \dots u_{i+n-1}$ be the factor of u of length n starting at position i. Justify that for all i, $u^{(i)} \neq av$. Deduce that each $u^{(i)}$ is a factor of length n of w with only one continuation.

For any factor u of length n of w which has exactly one continuation, it is possible to define $\mathbf{next}(u)$ as $w_{i+1} \dots w_{i+n}$ for any index i such that $u = w_i \dots w_{i+n-1}$. In other words, $\mathbf{next}(u)$ is the factor of length n of w starting at the second letter of u. (The fact that u has only one continuation justifies that this is a proper definition.)

(h) (*0.5 points*) Justify that for all $i \leq n$, $\mathbf{next}(u^{(i)}) = u^{(i+1)}$.

(i) (*1 point*) By the pigeon hole principle, argue that there exist $i < j$ such that $u^{(i)} = u^{(j)}$. And deduce that $\mathbf{next}(u^{(j-1)}) = u^{(i)}$.

(j) (*1 point*) Deduce that w is ultimately periodic. (A "proof by picture" using **next** can be enough.)

(k) (*1 point*) Conclude the proof that every Sturmian word is balanced.

Name: ______________________________ __

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