Exercise	Points	Max
1		2
2		3
3		5
4		5
5		7
6		4
Total		**26**

Authorized documents:
All your personal notes about this lecture and associated exercises.
Also, all documents posted on the websites of the lecture.
No books nor other lecture notes however.

Exercises are independent, and may be solved in the order you prefer.

Prefixes

Exercise 1

Let u, v and w be three finite words such that both u and v are prefixes of w.

Show that either u is a prefix of v or v is a prefix of u. *(2 points)*

Variants of equations

Exercise 2
Let a and b be two letters of a finite alphabet Σ. Let u be a word of $\Sigma^\star$.

Show (by induction on $|u|$) that if $ua = bu$, then $a = b$ and $u \in \{a\}^\star$. *(3 points)*

Periods

Exercise 3

In this exercise, questions (b), (c) and (d) can be solved independently of question (a).

Recall that, for any finite word v, its mirror image is denoted $\overrightarrow{v}$. That is to say, if $v = v_1 v_2 \dots v_n$, then $\overrightarrow{v} = v_n \dots v_2 v_1$.

(a) (*2 points*) For any finite word v, show that if p (with $1 \leq p \leq |v| - 1$) is a period of v, then p is also a period of $\overrightarrow{v}$.

Let u and v be two finite words. Let p be the smallest period of the word uv (or $p = |uv|$ if uv does not have a period). Also denote by p_u and p_v the smallest periods of u and v, respectively, if they exist (or $p_u = |u|$ (resp. $p_v = |v|$) if u (resp. v) does not have a period).

The goal of the remaining questions of this exercise is to show that $p \geq \max\{p_u, p_v\}$.

(b) (*1 point*) Prove that p is a period of u.

(c) (*1 point*) Similarly, prove that p is a period of v.

(d) (*1 point*) Conclude.

Property of the Thue-Morse morphism

Exercise 4

(a) (*3 points*) Let $X = \{ab, ba\}$. Show that for all $u \in X^\star$, we have $aua \notin X^\star$ and $bub \notin X^\star$.

Hint: You can do a proof using induction and/or contradiction.

Recall that the Thue-Morse morphism μ is defined on words over the alphabet $\Sigma = \{a, b\}$ by $\mu(a) = ab$ and $\mu(b) = ba$.

(b) (*2 points*) Deduce from the first question that, for any word w of $\Sigma^\star$, neither $a\mu(w)a$ nor $b\mu(w)b$ belong to $\mu(\Sigma^\star)$.

(For your information: This is a useful lemma in the proof that the Thue-Morse words has no overlapping factors.)

Complexity function of balanced words

Exercise 5

In this exercise, we work over the alphabet $\Sigma = \{a, b\}$.

Recall that an infinite word $w \in \Sigma^\omega$ is balanced when for all $n \geq 0$, and for any two factors u and v of w of length n, we have $\big||u|_a - |v|_a\big| \leq 1$.

Recall also that the complexity function of a word $w \in \Sigma^\omega$ is defined, for all $n \geq 0$, by $P_w(n) = |\mathcal{L}_n|$ where $\mathcal{L}_n$ is the set of factors of w of length n.

The goal of this exercise is to prove that if $w \in \Sigma^\omega$ is balanced, then for all $n \geq 0$, $P_w(n) \leq n + 1$.

Let w be a word of Σ^ω. Assume that $w \in \Sigma^\omega$ satisfies $P_w(n) \geq n+2$ for some $n \geq 0$. W.l.o.g., assume further that n is the smallest integer satisfying this condition.

(a) (*1 point*) First, justify that $n \geq 2$.

(b) (*1 point*) The assumption on n implies an upper bound on $|\mathcal{L}_{n-1}|$ and a lower bound on $|\mathcal{L}_n|$. What are they?

(c) (*1.5 points*) Deduce from these bounds that there exist two distinct words u and u' in $\mathcal{L}_{n-1}$ such that all four words ua, ub, $u'a$ and $u'b$ are in $\mathcal{L}_n$.

(d) (*1 point*) Justify that there exists a word $s \in \Sigma^\star$ such that as is a suffix of u and bs is a suffix of u', or vice versa.

(e) (*1.5 points*) Deduce that w is unbalanced.

(f) (*1 point*) Conclude.
(Hint: From question (a) to question (f), which statement have you proved? What is its contrapositive?)

Infinite words with no frequencies

Exercise 6

Recall that the frequency of a finite word u in a finite word w is $f_w(u) = \frac{|w|_u}{|w|}$

Recall also that the frequency of a finite word u in an infinite word w is defined as $\lim_{n\to\infty} \frac{|w_1 w_2 \ldots w_n|_u}{n}$, if this limit exists.

The goal of this exercise is to show that this limit does not always exist.

(a) (*1 point*) For all $i \geq 1$, let $w^{(i)} = a^{(2^i)} b^{(2^i)}$. Justify that the word $w = \lim_{i\to\infty} w^{(1)} w^{(2)} \ldots w^{(i)}$ exists.

(b) (*0.5 points*) For each $i \geq 1$, what is $f_{w^{(1)} w^{(2)} \ldots w^{(i)}}(a)$?

(c) (*1.5 points*) For each $i \geq 1$, what is $f_{w^{(1)} w^{(2)} \ldots w^{(i)} a^{(2^{i+1})}}(a)$? How does this quantity behave as i tends to infinity?

(d) (*1 point*) Conclude.

Name: ____________________ __

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