Homework II

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Exercise 1

(a) We have 3 cases:

- $x \in (A\cup B)^c \Longrightarrow \chi_A + \chi_B = 0 \,,\ \chi_{A\cup B} + \chi_{A\cap B} = 0$
- $x \in A\setminus B$ or $x \in B\setminus A \Longrightarrow \chi_A + \chi_B = 1 \,,\ \chi_{A\cup B} + \chi_{A\cap B} = 1$
- $x \in A\cap B \Longrightarrow \chi_A + \chi_B = 2,\ \chi_{A\cup B} + \chi_{A\cap B} = 2$

(b) Note that if $A\cap B = \phi$ then $\chi_{A\cap B}(x) \equiv 0$ and we conclude from (a).

Exercise 2

(a) $f_k \in T^{dec}$ & $f_k \longrightarrow \chi_I$

Since $\chi_I \in T$ then $f_k - \chi_I \in T^{dec}$. Moreove $f_k - \chi_I \longrightarrow 0$ a.e.
We can conclude from 10.2.10 that $\int f_k - \chi_I \longrightarrow 0$ and
so $\int f_k \longrightarrow \int \chi_I = \lambda_n(I)$.

(b) Let $f_n(x) = \begin{cases} n & x \in [0, \frac{1}{n}] \\ 0 & x \in (\frac{1}{n}, 1] \end{cases}$ then $f_n(x) \longrightarrow 0$ a.e. (see exercise sheet 1) but $\int f_n(x) = n \cdot \frac{1}{n} + 0 \cdot (1 - \frac{1}{n}) = 1 \not\longrightarrow 0$.

(c) We need to show that L is closed under addition and multiplication by a scalar (all the other properties are trivial).

- Let $f_1 - g_1 \,,\ f_2 - g_2 \in L$ then

$$(f_1 - g_1) + (f_2 - g_2) = \underbrace{(f_1 + f_2)}_{\in L^{inc}} - \underbrace{(g_1 - g_2)}_{\in L^{inc}} \in L \qquad (10.2.12)$$

• Let $\alpha \in \mathbb{R}$, $(f-g) \in L$ then

$$\alpha(f-g) = \underset{\substack{\cap \\ L^{inc}}}{\alpha f} - \underset{\substack{\cap \\ L^{inc}}}{\alpha g} \in L$$

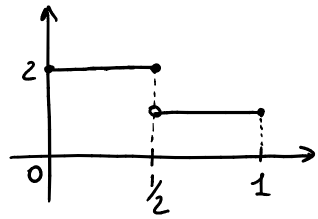
(simple proof)

Exercise 3

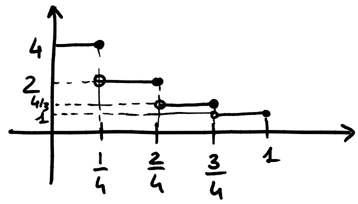
$$f : (0,1] \longrightarrow \mathbb{R}$$
$$x \longmapsto \frac{1}{x}$$

(a) We define $f_k = \sum_{j=1}^{2^k} \frac{2^k}{j} \chi_{((j-1)/2^k,\, j/2^k]}$.

Note that $f_1 = \sum_{j=1}^{2} \frac{2}{j} \chi_{(\frac{j-1}{2},\, \frac{j}{2}]}$



Note that $f_2 = \sum_{j=1}^{4} \frac{4}{j} \chi_{(\frac{j-1}{4},\, \frac{j}{4}]}$



Note that at every step we split every interval in two and we increase the value of the function in the left-hand side therefore $f_k \in T^{inc}$. Let $x \in (0,1]$ then

$$\frac{j-1}{2^k} < x \leq \frac{j}{2^k} \iff j-1 < 2^k x \leq j \implies x \in \left(\frac{\lceil 2^k x \rceil - 1}{2^k}, \frac{\lceil 2^k x \rceil}{2^k}\right]$$

and so $f_k(x) = 2^k / \lceil 2^k x \rceil$.

Note that $\underbrace{\frac{2^k}{2^k x+1}}_{\downarrow k\to\infty \quad 1/x} \leq \frac{2^k}{\lceil 2^k x\rceil} \leq \underbrace{\frac{2^k}{2^k x-1}}_{\downarrow k\to\infty \quad \frac{1}{x}}$ and so $f_k(x) \longrightarrow \frac{1}{x}$.

(b) We start by computing

$$\int f_k = \sum_{j=1}^{2^k} \frac{2^k}{j} \underbrace{\lambda_1\left(\left(\frac{j-1}{2^k}, \frac{j}{2^k}\right]\right)}_{1/2^k} = \sum_{j=1}^{2^k} 1/j \xrightarrow[k\to+\infty]{} +\infty$$

$\Rightarrow \int f_k$ is unbounded. (**)

Now take any sequence $g_\ell \in T^{inc}$ s.t. $g_\ell \xrightarrow{a.e.} 1/x$

We set

$$f_{k,\ell} = \min(f_k, g_\ell) \qquad f_{k,\ell} \xrightarrow{\ell\to\infty} f_k$$

Since f_k and g_ℓ are step functions then $f_{k,\ell}$ is also a step function. Note that

$$\lim_{\ell\to\infty} f_{k,\ell} = \lim_{\ell\to\infty} \min(f_k, g_\ell) = \min\left(f_k, \tfrac{1}{x}\right) = f_k \quad \left(\text{since } f_k \uparrow \tfrac{1}{x}\right)$$

and so by thm 10.2.10 (used as in (a)) we have

$$\forall k \qquad \int f_{k,\ell} \xrightarrow[\ell\to\infty]{} \int f_k \iff \int f_{k,\ell} - f_k \xrightarrow[\ell\to\infty]{} 0 \quad (*)$$

Clearly, $g_\ell \geq f_{k,\ell} = \min(f_k, g_\ell) \Rightarrow \int g_\ell \geq \int f_{k,\ell}$

Therefore

$$\int g_\ell \geq \int f_{k,\ell} = \int f_{k,\ell} - f_k + \int f_k \quad (*)$$

Now fix $\varepsilon > 0$ small and $M > 0$ big.

Then by $(**)$ we can chose $\bar{k}$ large enough so that $\int f_{\bar{k}} \geq M$

By $(*)$ we can chose $\bar{\ell}$ large enough such that $\int f_{\bar{k},\bar{\ell}} - f_{\bar{k}} \geq -\varepsilon$.

We can conclude from $(*)$ that

$$\int g_\ell \geq M - \varepsilon, \text{ for all } \ell \geq \bar{\ell},$$

concluding the proof of the fact that $\int g_\ell$ is unbounded.

Exercise 4

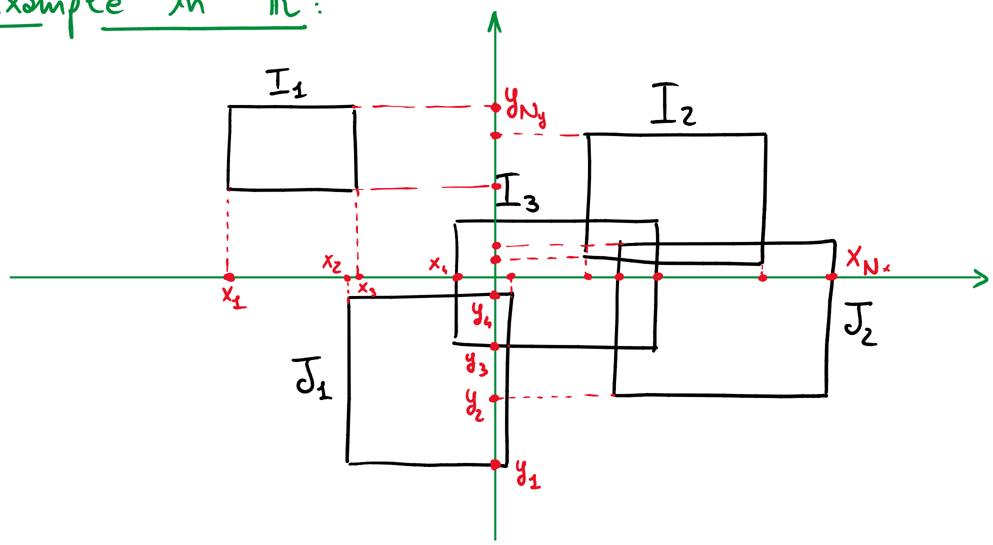
I_k, $k = 1, \dots, N$

J_ℓ, $\ell = 1, \dots, M$

intervals of $\mathbb{R}^n$ (i.e. $I_i = I_i^1 \times \dots \times I_i^n$ with I_i^j intervals of $\mathbb{R}^n$.)

We want to construct a refinement that is disjoint.

Example in $\mathbb{R}^2$:



We consider the intervals $L_{i,j} = (x_i, x_{i+1}) \times (y_j \times y_{j+1}) \quad \forall\, i \leq N_x,\ j \leq N_y$.

The collection $\{ L_{i,j} \mid L_{i,j} \text{ is contained in some } I_k \text{ or } J_\ell \}$

is a disjoint refinement.

In general, first of all note that the fact that we are dealing with two sequeces I_k, J_ℓ does not have influence.

So we just assume that $(K_i)_{i=1}^N$ is the given collection of intervals.

$$\forall i \qquad K_i = K_i^1 \times \dots \times K_i^n \quad \text{with } (K_i^j)_j \text{ intervals of } \mathbb{R}$$

For $j \le n$, let $(x_\ell^j)_{\ell=1}^{N_j}$ be the collection of boundary coordinates of $(K_i^j)_{i=1}^N$

Then

$$\left\{ L_{\ell_1,\dots,\ell_n} = (x_{\ell_1}^1, x_{\ell_1+1}^1) \times \dots \times (x_{\ell_n}^n, x_{\ell_n+1}^n) \ \forall (\ell_1,\dots,\ell_n) \in \underbrace{[N_1]}_{\{1,\dots,N_1\}} \times \dots \times \underbrace{[N_n]}_{\{1,\dots,N_n\}} \,\middle|\, L_{\ell_1,\dots,\ell_n} \subseteq K_i \text{ for some } i \right\}$$

is a disjoint refinement.

Exercise 5

$$A = \left\{ \sum_{n=1}^{\infty} a_n 3^{-n} \,\middle|\, a_n \in \{0,2\} \right\}$$

$$\phi(x) = \sum_{n=1}^{\infty} a_n 2^{-(n+1)} \qquad \text{when} \quad x = \sum_{n=1}^{\infty} a_n 3^{-n}$$

(a) We can identify $x \in [0,1]$ with the corresponding seq. $(a_n)_n$:

$$x \overset{3}{\longleftrightarrow} (a_n)_n$$

Note that if $x \overset{3}{\longleftrightarrow} (a_n)_n$ & $y \overset{3}{\longleftrightarrow} (b_n)_n$

$$x \leq y \iff (a_n)_n \ll (b_n)_n$$

↳ lexicograph order

We have also the relation $\overset{2}{\longleftrightarrow}$ similarly defined. Note that if $x \overset{3}{\longleftrightarrow} (a_n)_n$

$$\phi(x) = y \iff y \overset{2}{\longleftrightarrow} (a_n/2)_n$$

<u>ϕ is non-decreasing</u>:

Take $(a_n)_n \ll (b_n)_n$ in A. Obviously $\left(\frac{a_n}{2}\right) \ll \left(\frac{b_n}{2}\right)$ and so $\phi((a_n)_n) \leq \phi((b_n)_n)$.

<u>ϕ is surjective</u>:

Let $y \in [0,1]$ s.t. $y \overset{2}{\longleftrightarrow} (a_n)_n$

Take $x \overset{3}{\longleftrightarrow} (2a_n)_n$. Then $\phi(x) = y$.

<u>ϕ is continuous</u>:

We show that the preimage of a closed set is closed. It is enough to prove that for closed intervals since they form a basis.

Let $[x,y] \subseteq [0,1]$ and we compute the pre-image:

If $x \overset{2}{\longleftrightarrow} (a_n)_n$, $y \overset{2}{\longleftrightarrow} (b_n)$ then

$$[x,y] = \left\{ (s_n)_n \in \{0,1\}^{\mathbb{N}} \,\middle|\, (a_n)_n \ll (s_n)_n \ll (b_n)_n \right\}$$

and since ϕ is non-decreasing and surjective

$$\phi^{-1}([x,y]) = \left\{ (s_n)_n \in \{0,2\}^{\mathbb{N}} \,\middle|\, (2a_n) \ll (s_n)_n \ll (2b_n)_n \right\}.$$

and since ϕ is non-decreasing and bijective

$$\phi^{-1}([x,y]) = \left\{ (s_n)_n \in \{0,2\}^{\mathbb{N}} \;\middle|\; (2a_n)_n \ll (s_n)_n \ll (2b_n)_n \right\}.$$

Therefore $\phi^{-1}([x,y]) = [x',y'] \cap A$ with $x' \overset{3}{\longleftrightarrow} (2a_n)_n$, $y' \overset{3}{\longleftrightarrow} (2b_n)_n$

and since this is a closed set we conclude the proof.

(b) Simply take $\phi^{+}(x) = 0 \quad \forall x \in [0,1] \setminus A$.

(c) The previous extension is <u>NOT</u> continuous.

Let $x \in [0,1] \setminus A$. $x \overset{3}{\longleftrightarrow} (a_n)_n$. There exists a minimal N_x s.t. $a_{N_x} = 1$ (otherwise $x \in A$). Note that

$$x_- \leq x \leq x_+$$

where

$$x_- = (a_1, \dots, a_{N_x - 1}, 0, 2, 2, 2, \dots)$$
$$x_+ = (a_1, \dots, a_{N_x - 1}, 2, 0, 0, 0, \dots)$$

Note that $x_-, x_+ \in A$ & $\phi(x_-) = \phi(x_+) = v$.
Therefore extending ϕ^{+} in x by $\phi^{+}(x) := v$ we have a continuous extension of ϕ.