Homework III

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Exercise 1

$f, g \in L^1(\mathbb{R}^n, \mathbb{R})$

(a) Show that $|f| \in L^1(\mathbb{R}^n, \mathbb{R})$:

Since $f \in L^1(\mathbb{R}^n, \mathbb{R})$ then $f = f_1 - f_2$ for some $f_1, f_2 \in L^{inc}$.

Since

$$\max\{f_1, f_2\} - \min\{f_1, f_2\} = |f_1 - f_2| = |f|,$$

it suffices to show that $\max\{f_1, f_2\}, \min\{f_1, f_2\} \in L^{inc}$.

Take $(a_k)_k, (b_k)_k \in T^{inc}$ so that $a_k \xrightarrow{a.e.} f_1$, $b_k \xrightarrow{a.e.} f_2$

Then $\max\{a_k, b_k\}$ & $\min\{a_k, b_k\}$ are also step functions, monotonically increasing converging a.e. to $\max\{f_1, f_2\}$ & $\min\{f_1, f_2\}$ resp.

(b) Show that $\min\{f, g\}$ & $\max\{f, g\}$ are Leb. integrable:

Since

$$\max\{f, g\} + \min\{f, g\} = f + g$$

and

$$\max\{f, g\} - \min\{f, g\} = |f - g|$$

then

$$\max\{f, g\} = \frac{f + g + |f - g|}{2} \in L^1 \quad \text{(vector space)}$$

and

$$\min\{f, g\} = \frac{f + g - |f - g|}{2} \in L^1 \quad \text{(vector space)}$$

Exercise 2

$f : [0,1] \longrightarrow \mathbb{R}$ strictly increasing function. Show that $f \in L^1([0,1])$.

<u>Solution</u>: Let D be the set of discontinuity points of f.
At every point $x \in D$, due to monotonicity, there exists both

$$L_x = \lim_{y \to x^-} f(y)$$

$$R_x = \lim_{y \to x^+} f(y)$$

Since f is discontinuous at x & strictly increasing then

$$L_x < R_x$$

Therefore, $\exists \; q_x \in \mathbb{Q} \cap (L_x, R_x)$. Moreover, if $x, y \in D$ and $x < y$ then $q_x < q_y$. Hence there exists an injection from D to $\mathbb{Q}$. So D is countable and $\lambda([0,1]) = 0$.
Therefore $f \in L^1([0,1])$.

Exercise 3

$$f : [0,1] \times [0,1] \longrightarrow \mathbb{R} \quad \text{s.t.} \quad f(x) = \frac{1}{\|x\|} \chi_{\{(0,0)\}^c}$$

Construct a sequence of step functions such that $(f_k)_k \in T^{inc}$ so that $f_k \xrightarrow{a.e.} f$ as $k \to \infty$, & show that $\lim_{k \to \infty} \int f_k$ exists and it is bounded.

<u>Solution</u>: Define $I^k_{i,j} := \left[\frac{i-1}{2^k}, \frac{i}{2^k}\right] \times \left[\frac{j-1}{2^k}, \frac{j}{2^k}\right] \quad \forall \, i,j \in \{1, \dots, 2^k\}$
Then define

$$f_k(x,y) = \sum_{i,j=1}^{2^k} \frac{1}{\|(\frac{i}{2^k}, \frac{j}{2^k})\|} \chi_{I^k_{ij}}(x,y)$$

$f_k \xrightarrow{a.e.} f$ and $f_k \in T^{inc}$ (same proof as previous ex. sheet)

Now

$$(*) \quad \int f_k = \sum_{i,j=1}^{2^k} \frac{1}{\|(i/2^k, j/2^k)\|} \left(\frac{1}{2^k}\right)^2 = \sum_{i,j=1}^{2^k} \frac{1}{4^k \sqrt{\frac{i^2+j^2}{4^k}}} = \sum_{i,j=1}^{2^k} \frac{1}{2^k \sqrt{i^2+j^2}}$$

Since $f_k \in T^{inc}$ and $f_k \geq 0$ then $(\int f_k)_k$ is increasing and so we just need to show that $(\int f_k)_k$ is bounded.

We have from $(*)$ that

$$(**) \quad \int f_k = \sum_{i,j=1}^{2^k} \frac{1}{2^k \sqrt{i^2+j^2}} \leq 2^{1-k} \sum_{i,j=1}^{2^k} \frac{1}{i+j}$$

$$\frac{1}{\sqrt{x^2+y^2}} \leq 2 \frac{1}{x+y} \qquad x, y > 0$$

Now we estimat $\displaystyle\sum_{k=n}^{m} \frac{1}{k} \leq \frac{1}{n} + \int_n^m \frac{1}{t}\, dt = \frac{1}{n} + \log\left(\frac{m}{n}\right)$

Therefore from $(**)$ we have that

$$\int f_k \leq 2^{1-k} \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} \frac{1}{i+j} \leq 2^{1-k} \sum_{i=1}^{2^k} \frac{1}{i+1} + \underbrace{\log\left(\frac{i+2^k}{i+1}\right)}_{\hookrightarrow \text{ decreasing in } i}$$

$$\leq 2^{1-k} \Bigg(\underbrace{\frac{1}{2} + \log\left(\frac{2^k+1}{2}\right)}_{\text{as before}} + \log\left(\frac{1+2^k}{2}\right) + \int_1^{2^k} \log\left(\frac{t+2^k}{t+1}\right) dt \Bigg)$$

$$\leq 2^{1-k} \left(2^{-1} + \log\left(\frac{4^k+1+2^{k+1}}{4}\right)\right) + C\, 2^{-k} \left(t(\log t - 1)\Big|_{2^k+1}^{2^{k+1}} - t(\log t - 1)\Big|_2^{2^k} \right)$$

$$\leq C' + \log\left(\frac{4^{k+1}}{(2^k+1)\, 2^k}\right) \leq C''$$

Exercise 4

(a) $$\int \frac{y^2-x^2}{(x^2+y^2)^2}\,dy = -\frac{y}{x^2+y^2} \quad \& \quad \int \frac{y^2-x^2}{(x^2+y^2)^2}\,dx = \frac{x}{x^2+y^2}$$

So $$\int_0^1\int_0^1 \frac{y^2-x^2}{(x^2+y^2)^2}\,dy\,dx = \int_0^1 \left(-\frac{y}{x^2+y^2}\right)\Big|_{y=0}^{1} dx = \int_0^1 -\frac{1}{1+x^2}\,dx =$$

$$= -\operatorname{arctg}(x)\Big|_0^1 = -\frac{\pi}{4}$$

$$\int_0^1\int_0^1 \frac{y^2-x^2}{(x^2+y^2)^2}\,dx\,dy = \int_0^1\left(\frac{x}{x^2+y^2}\right)\Big|_{x=0}^{1} dy = \int_0^1 \frac{1}{1+y^2}\,dy = \frac{\pi}{4}$$

Therefore $\frac{y^2-x^2}{(x^2+y^2)^2}$ is not Leb. integ. (by Fubini).

(b) $$\int \frac{xy}{(x^2+y^2)^2}\,dy = -\frac{x}{2(x^2+y^2)} \quad \& \quad \int \frac{xy}{(x^2+y^2)^2}\,dx = -\frac{y}{x^2+y^2}$$

Then we obtain by standard computations that
$$\int_0^1\int_0^1 \frac{xy}{(x^2+y^2)^2}\,dy = +\infty \quad \&$$
$$\int_0^1\int_0^1 \frac{xy}{(x^2+y^2)^2}\,dx = +\infty$$

and so by Fubini the function is NOT Leb. integrab.

Exercise 5

$(q_k)_{k\in\mathbb{N}_0}$ a seq. of $\mathbb{Q}\cap[0,1]$, with $q_0 = 1$. $\forall\, \varepsilon>0,\ k\in\mathbb{N}_0$

$$I_k := \left(q_k - \frac{\varepsilon}{2}10^{-k},\ q_k + \frac{\varepsilon}{2}10^{-k}\right)$$

Show that $\exists\ \varepsilon>0$ so that $\bigcup_{k\in\mathbb{N}_0} I_k \not\supseteq [0,1]$.

<u>Solution</u>:

We prove the claim for $\varepsilon = \frac{1}{10}$. Let $q_k = 0.q_1^k q_2^k q_3^k$ be the decimal notation of q_k. And take $x = 0.x_1x_2x_3\ldots$

The condition

$$x \notin \left(q_k - \frac{10^{-k-1}}{2} , q_k + \frac{10^{-k-1}}{2} \right)$$

is true if $|x_k - q_k^k| > 2$. Indeed

$$x \notin \left(q_k - \frac{10^{-k-1}}{2} , q_k + \frac{10^{-k-1}}{2} \right) \Longleftrightarrow |x - q_k| \geq \frac{10^{-k-1}}{2}$$

$$\Longleftrightarrow 10^{k+1} |x - q_k| \geq \frac{1}{2} \Longleftrightarrow \left| x_1 \ldots x_{k+1}, x_{k+2}\ldots - q_1^k \ldots q_{k+1}^k, q_{k+2}^k \right| \geq \frac{1}{2}$$

and the last condition is satisfied if $|x_k - q_k^k| > 2$.
(it is also satisfied if for instance $|x_{k+1} - q_{k+1}^k| \geq 1$)
Hence setting

$$x_k := \begin{cases} 0 & \text{if } q_k^k \geq 5 \\ 9 & \text{if } q_k^k < 5 \end{cases} \qquad \forall k \in \mathbb{N}_0$$

Such x is non included in any I_k and so $[0,1] \not\subseteq \bigcup_k I_k$. $\square$