# Homework IV

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## Exercise 1

We want to show that if $f \in L^1(\mathbb{R}^2)$ then

$$\int_{[0,1]} \int_{[0,x]} f \, dy \, dx = \int_{[0,1]} \int_{[y,1]} f \, dx \, dy.$$

Note that

$$\int_{[0,1]} \int_{[0,x]} f \, dy \, dx = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y) \underbrace{\chi_{[0,1]}(x) \chi_{[0,x]}(y)}_{\substack{\shortparallel \\ \chi_{\{0 \le x \le 1,\, 0 \le y \le x\}}(x,y) \\ \shortparallel \\ \chi_{\{0 \le y \le x \le 1\}}(x,y) = \chi_{[0,1]}(y) \chi_{[y,1]}(x)}} dy \, dx = \Delta$$

Since $\underset{\substack{\cup \\ L^1}}{f}(x,y) \underbrace{\chi_{[0,1]}(x) \chi_{[0,x]}(y)}_{\text{step-function}} \in L^1(\mathbb{R}^2)$ then by Fubini

$$\Delta = \int_{\mathbb{R}} \int_{\mathbb{R}} f \chi_{[0,1]}(y) \chi_{[y,1]}(x) \, dx \, dy = \int_{[0,1]} \int_{[y,1]} f(x,y) \, dx \, dy.$$

## Exercise 2

$f \in L^1(\mathbb{R}^n, \mathbb{R})$ & $\int_{\mathbb{R}^n} f = 0 \Longrightarrow f = 0$ a.e.

↪ non-negative

Solution:

Consider the set $E_n = \left\{ x \in \mathbb{R}^n \mid f(x) \ge \frac{1}{n} \right\}$ and set

$E = \bigcup_{n \ge 0} E_n = \{ x \in \mathbb{R}^n \mid f(x) > 0 \}$

$$0=\int_{\mathbb{R}^n} f(x)\,dx = \int_E f(x)\,dx + \underbrace{\int_{\mathbb{R}^n\setminus E} f(x)\,dx}_{=0} \underset{E \supseteq E_n}{\geq} \int_{E_n} f(x)\,dx \geq \lambda(E_n)\frac{1}{n}$$

$$\Rightarrow \lambda(E_n)=0 \quad \forall n \quad \Rightarrow \lambda(E)=0 \Rightarrow f=0 \quad \text{a.e.}$$

## Exercise 5

- $f \in L^1(\mathbb{R})$, $\xi \in \mathbb{R}$

  Show that $g_\xi(x) := f(x)\cos(\xi x) \in L^1(\mathbb{R})$

  Solution:

  Since $\cos(\xi\,\cdot)$ is bounded and continuos, the $\exists$ a seq. of step functions $C_k(\cdot)$ s.t. $C_k \uparrow \cos(\xi\,\cdot)$. Note also that since $\cos(\xi\,\cdot)$ is bounded $\exists$ $C>0$ s.t. $|C_k(x)| \leq C$ $\forall x \in \mathbb{R}$. Therefore we can conclude that

  $$f(x)\,C_k(x) \longrightarrow f(x)\cos(\xi x)$$

  and

  $$|f\,C_k(x)| \leq |f|\,C \in L^1$$

  Then by dominated convergence theorem we can conclude that $f(x)\cos(\xi x)$ is Leb. integrable.

- We define

  $$K : L^1(\mathbb{R}) \longrightarrow \mathcal{C}(\mathbb{R})$$
  $$f \longmapsto K(f) \quad \text{s.t.} \quad K(f)(\xi) = \int f(x)\cos(\xi x)\,dx$$

  We show that $K$ is well-def:

  $\oplus$ $\forall\, \xi \in \mathbb{R}$, $K(f)(\xi)$ exists and is finite.

**Solution:** Since $K(f)(\varepsilon) = \int g_\varepsilon(x)\,dx$ and we proved that $g_\varepsilon(x) \in L^1$ we can conclude that $K(f)(\varepsilon)$ exists and is finite.

$\oplus$ $K(f) \in \mathcal{C}(\mathbb{R})$ :

**Solution:** We show that $\forall$ seq. $\xi_n \longrightarrow \xi_0$ then

$$K(f)(\xi_n) \longrightarrow K(f)(\xi_0)$$

We have that

$$\lim_{n\to\infty} K(f)(\varepsilon_n) = \lim_{n\to\infty} \int f(x)\cos(\varepsilon_n x)\,dx \overset{(?)}{=} \int \lim_{n\to\infty} f(x)\cos(\varepsilon_n x)\,dx \overset{\text{by cont. of cos}}{=} K(f)(\varepsilon_0)$$

It remains to justify (?) :

Note that

- $f_n(x) := f(x)\cos(\varepsilon_n x) \longrightarrow f(x)\cos(\varepsilon_0 x)$.
- $|f_n(x)| \leq |f(x)| \in L^1$

Then (?) is justified by the dom. conv. theorem.

## Exercise 3

$I \subseteq \mathbb{R}$ interval , $f : I \longrightarrow \mathbb{R}$ continuous. $G = \{(x, f(x)) ;\ x \in I\}$

(a) $I$ compact $\Rightarrow \lambda_2(G) = 0$.

**Solution:** Since $f$ is cont. on a compact, the $f$ is unif. cont.

Hence $\forall \varepsilon > 0$, $\exists \delta$ s.t. if $|x-y| < \delta$ then $|f(x)-f(y)| < \varepsilon$

We take a subdivision of $I = [a,b]$ by $I_j^\delta = [a + j\delta, a + (j+1)\delta)$ $\forall j$ s.t. $j \in \{0, 1, \dots, \lceil \frac{b-a}{\delta} \rceil - 1\}$. Then

$$G \subseteq \bigcup_j I_j^\delta \times [f(a+j\delta) - \varepsilon, f(a+j\delta) + \varepsilon]$$

$$\Rightarrow \lambda_2(G) \leq \left\lceil \frac{b-a}{\delta} \right\rceil \cdot \delta \cdot 2\varepsilon \leq \left(\frac{b-a}{\delta}+1\right) \delta \cdot 2\varepsilon \xrightarrow[\varepsilon \to 0]{} 0$$

(b) $I = \mathbb{R} \Rightarrow \lambda_2(G) = 0$.

Solutions: Write $\mathbb{R} = \bigcup_{k \geq 1} [-k, k]$. Then we can write the graph $G$ as a countable union of zero-measure sets.

## Exercise 4

(a) $f(x) = -x^{-1/2}$. Construct $g_n, h_n \in L^{inc}$ so that $f = g_n - h_n$ & $\int h_n < \frac{1}{n}$.

Solution:

$$f = \underbrace{f \cdot \chi_{[\frac{1}{k}, 1]}}_{g_n} - \underbrace{\left(-f \cdot \chi_{(0, \frac{1}{k})}\right)}_{h_n \in L^{inc}}$$

$g_n \in L^{inc}$ (since cont. on a compact) (example 10.2.9)

$$\int -f \chi_{(0, 1/k)} = \int_0^{1/k} \frac{1}{\sqrt{x}} dx = \left[2\sqrt{x}\right]\Big|_{x=0}^{1/k} = 2\sqrt{1/k} < \frac{1}{n}$$

for k big enough

(b) $f : [1, \infty)$ def. by $f(x) = (e^x - 1)^{-1}$. $f \in L^1$?

Solution: We set $f_n = f \chi_{[1,n]}$. Since $f_n \uparrow f$, and $\int f_n = \int_0^n f = \log\left(\frac{e^n - 1}{e^n}\right) + \log\left(\frac{e}{e-1}\right) \longrightarrow \log\left(\frac{e}{e-1}\right)$ then

Leb int = Riem int for cont. fct on compact sets.

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$$\int f = \lim \int f_n = \log\left(\frac{e}{e-1}\right)$$