Homework 0

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Exercise

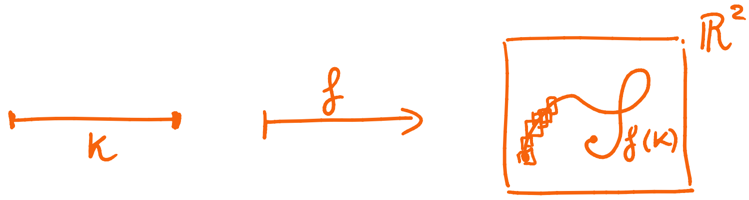
Let $U \subseteq \mathbb{R}^m$, $f \in \mathcal{C}^1(U, \mathbb{R}^n)$, $m < n$. $K \subseteq U$ and K compact.

Show that $f(K)$ has zero measure in $\mathbb{R}^n$.

Does the same holds for $f \in \mathcal{C}^0(U, \mathbb{R}^n)$? [Look at the video]

Solution:

IDEA: If $m=1$ & $n=2$ the situation is the following:



We can "cover" $f(K)$ with cubes of edge-size $\sim x$ and we need $\sim \frac{|K|}{x}$ cubes. The total area will be $x^2 \frac{|K|}{x} \xrightarrow[x \to 0]{} 0$

Since $f \in \mathcal{C}^1$, let L be the Lipshitz constant. Note that if $Q \subseteq U$ is a cube of edge-size δ then

$f'(Q)$ is contained in a cube of edge-size $C \cdot L \cdot \delta$ universal constant

Indeed if $x, y \in Q$ then

$$|f(x) - f(y)| \leq L\,|x-y| \leq L \cdot C \cdot \delta$$

Now chose a cube Q of U s.t. $\overset{\text{compact}}{K} \subseteq Q$

Define $\{Q^j_\ell\}_{\ell=1}^{(2^j)^m}$ to be the 2^j-th section of Q.

Example: If $m=3$ the the 2-nd section of Q is given by the $(2^1)^3 = 8$ red cubes in this picture:

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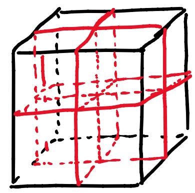

Note that if the edge size of Q is D then the edge-size of Q_ℓ is $D/2^j$. Hence we can cover $f(K)$ with $\{f(Q_\ell^j)\}_{\ell=1}^{2^{mj}}$, whose edge-size is bounded by $\frac{L \cdot C \cdot D}{2^j}$.

The total volume of $\{f(Q_\ell^j)\}_{\ell=1}^{2^{mj}}$ is then bounded by

$$2^{mj}\left(\frac{L\cdot C\cdot D}{2^j}\right)^n = (LCD)^n\, 2^{j\overbrace{(m-n)}^{<0}} \xrightarrow[j\to\infty]{} 0$$

Therefore we can conclude that $f(K)$ has zero-measure.