Homework I

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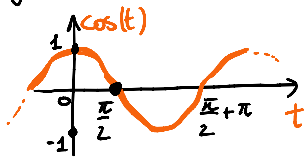
Exercise 1

(a) $\displaystyle\int_{\frac{\pi}{2}}^{\infty} \frac{\cos(t)}{t}\, dt$ exists?

Let $\displaystyle a_n := \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+n\pi} \frac{\cos(t)}{t}\, dt$ and $\displaystyle b_j := \int_{\frac{\pi}{2}+(j-1)\pi}^{\frac{\pi}{2}+j\pi} \frac{\cos(t)}{t}\, dt$.

Note that $\displaystyle a_n = \sum_{j=1}^{n} b_j$. We want to show that $\lim_{n\to\infty} a_n$ exists.

Note that since 

then

$b_j < 0$, for all j odd & $b_j > 0$ for all j even.

Therefore, in order to show that a_n converges, it is enough by Leibnitz criterion to show that

(a) $|b_j|$ decreases monotonically

(b) $\lim_{j\to\infty} |b_j| = 0$

$$\left.\begin{array}{l} t_1 \in \left(\frac{\pi}{2}+(j-1)\pi, \frac{\pi}{2}+j\pi\right) \\ t_2 \in \left(\frac{\pi}{2}+j\pi, \frac{\pi}{2}+(j+1)\pi\right) \end{array}\right\} \Rightarrow \frac{1}{t_1} \geq \frac{1}{t_2}$$

Proof of (a):

$$|b_j| = \int_{\frac{\pi}{2}+(j-1)\pi}^{\frac{\pi}{2}+j\pi} \left|\frac{\cos(t)}{t}\right| dt \geq \int_{\frac{\pi}{2}+j\pi}^{\frac{\pi}{2}+(j+1)\pi} \left|\frac{\cos(t)}{t}\right| dt = |b_{j+1}|$$

Proof of (b):

$$|b_j| = \int_{\frac{\pi}{2}+(j-1)\pi}^{\frac{\pi}{2}+j\pi} \left|\frac{\cos(t)}{t}\right| dt \leq \int_{\frac{\pi}{2}+(j-1)\pi}^{\frac{\pi}{2}+j\pi} \frac{1}{\frac{\pi}{2}+(j-1)\pi}\, dt = \frac{\pi}{\frac{\pi}{2}+(j-1)\pi} \xrightarrow[j\to\infty]{} 0.$$

(b) $\int_{\frac{\pi}{2}}^{\infty} \left| \frac{\cos(t)}{t} \right| dt$ exists ?

Note that

$$b_j := \int_{\frac{\pi}{2}+(j-1)\pi}^{\frac{\pi}{2}+j\pi} \left|\frac{\cos(t)}{t}\right| dt \geq \int_{\frac{\pi}{2}+(j-1)\pi}^{\frac{\pi}{2}+j\pi} \frac{|\cos(t)|}{\frac{\pi}{2}+j\pi}\, dt = \frac{2}{\frac{\pi}{2}+j\pi} \geq \frac{c}{j}$$

$\longrightarrow \sum_{j=1}^{\infty} b_j \geq \sum_{j=1}^{\infty} \frac{1}{j} = \infty$. We conclude noting that

$$\int_{\frac{\pi}{2}}^{\infty} \left|\frac{\cos(t)}{t}\right| dt = \sum_{j=1}^{\infty} b_j = +\infty .$$

$\square$

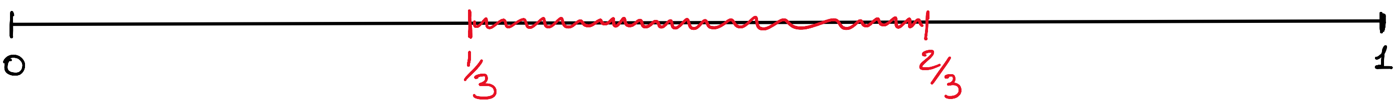
Exercise 2

(a) The set $\mathbb{Q}$ is a countable union of singletons, therefore it is of measure zero (because a countable union of zero-measure sets has zero-measure).

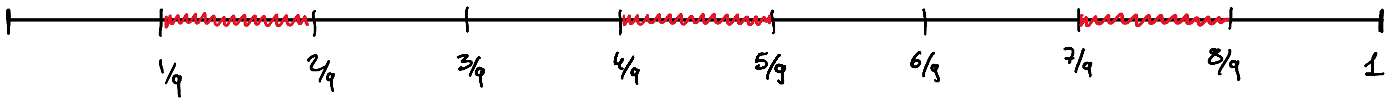
We know that $\mathbb{R}$ has no zero-measure therefore $\mathbb{R} \setminus \mathbb{Q}$ has no zero-measure too (otherwise $\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q}$ would have zero-measure.

(b) We first look at the sets $\left(\frac{1}{3^n}\left((1,2) + 3\mathbb{Z}\right)\right) \cap [0,1]$:

for $n=1$: We have the set $\left(\left(\frac{1}{3}, \frac{2}{3}\right) + \mathbb{Z}\right) \cap [0,1]$:

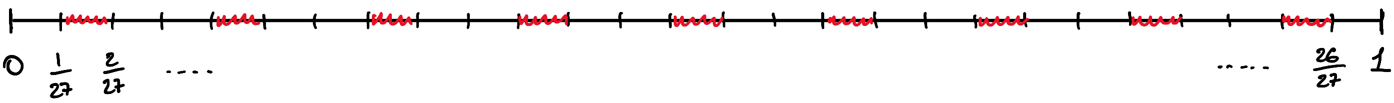


for $n=2$: We have the set : $\left(\left(\frac{1}{9}, \frac{2}{9}\right) + \frac{\mathbb{Z}}{3}\right) \cap [0,1]$:



$1/9$ $2/9$ $3/9$ $4/9$ $5/9$ $6/9$ $7/9$ $8/9$ 1

for $n=3$: We have the set $\left(\left(\frac{1}{27}, \frac{2}{27}\right) + \frac{\mathbb{Z}}{9}\right) \cap [0,1]$:



In particular, we have that

$$A_1 = [0, 1/3] \cup [2/3, 1]$$

$$A_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$$

By construction, at every step we remove $\frac{1}{3}$ of the remaing set, so we have $\lambda_1(A_n) = 2/3\, \lambda_1(A_{n-1})$. Since $\lambda_1(A_1) = \frac{2}{3}$ we obtain that

$$\lambda_1(A_n) = (2/3)^n.$$

(c) Writing numbers in base 3 we have:

$$[0,1] = \left\{ \sum_{j=1}^{\infty} a_j 3^{-j} : a_j \in \{0,1,2\} \right\}$$

Similarly (look at the pictures above) we have that

$$\left(\frac{1}{3^n}((1,2) + 3\mathbb{Z})\right) \cap [0,1] = \left\{ \sum_{j=1}^{\infty} a_j 3^{-j} : a_j \in \{0,1,2\}, a_n = 1 \right\}$$

Therefore, we have that

$$A_n := \left\{ \sum_{j=1}^{\infty} a_j 3^{-j} : a_j \in \{0,2\} \ \forall j \leq n,\ a_j \in \{0,1,2\} \ \forall j > n \right\}$$

So the intersection of all of them must be:

$$A = \left\{ \sum a_i 3^{-j} : a_i \in \{0,2\} \ \forall i \right\}$$

$$A = \left\{ \sum_{j=1}^{\infty} a_j \, 3^{-j} \; : \; a_j \in \{0,2\} \;\; \forall j \right\}$$

Its cardinality is $2^{|\mathbb{N}|}$ since we have a bijection between A and $\{0,2\}^{\mathbb{N}}$.

(d) $A \subseteq A_n \quad \forall n$. The measure of A_n is $(2/3)^n$

Therefore the measure of A must be zero, since $(2/3)^n \xrightarrow[n \to \infty]{} 0$.

Exercise 3

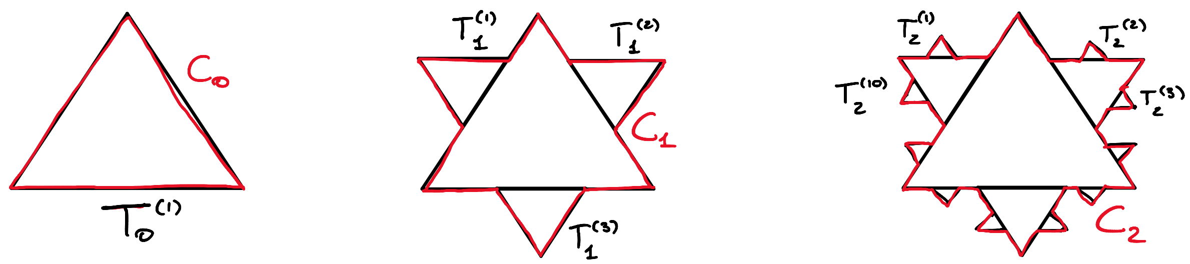
We know that a function $f : [a,b] \longrightarrow \mathbb{R}$ is Riemann integrable if and only if f is bounded and almost everywhere continuous.

The function d is discontinuous at every point of $[0,1]$ therefore it is <u>NOT</u> Riemann integrable.

From exercise 2, we know that $\mathbb{Q}$ has measure zero.

Since $\{ x \in [0,1] \mid d(x) = 0 \} = [0,1] \setminus \mathbb{Q}$ then we can conclude that $d = 0$ almost everywhere.

Exercise 4



(a) Assuming that $\lambda_1(C_0) = 1$, then, at each step, every segment of the curve C_{n-1} gives 4 segments of length $\frac{1}{3}$ of of the curve C_{n-1} gives 4 segments of length $\frac{1}{3}$ of the initial length of the segment. Therefore

$$\begin{cases} \lambda_1(C_n) = \frac{4}{3}\lambda(C_{n-1}) \\ \lambda_1(C_0) = 1 \end{cases} \Rightarrow \lambda_1(C_n) = \left(\frac{4}{3}\right)^n$$

Therefore the length of the Koch snowflake is $\lim_{n\to\infty} (4/3)^n = +\infty$.

(b) Note that (see the pictures above) the snowflake was constructed starting with a triangle T_0, then adding 3 triangles of type T_1, then 10 triangles of type T_2 and so on.

Let $\ell_i :=$ # of triangles T_i that we add at the step i.

Obviously $\ell_i < \infty$, $\forall i$. Moreover denote by

$$\mathcal{T}_i = \bigcup_{j=1}^{\ell_i} T_i^{(j)} \quad \left(\text{i.e. the union of the } \ell_i \text{ triangles of type } T_i\right)$$

Note that $\lambda_2(T_i^{(j)}) = 0 \quad \forall i,j$ and so

$$\lambda_2(\mathcal{T}_i) = 0 \quad (\text{since finite union of zero-meas. sets})$$

Noting that the Koch snowflake γ satisfies

$$\gamma \subseteq \bigcup_{i=1}^{\infty} \mathcal{T}_i$$

we can conclude that $\lambda_2(\gamma) = 0$ (using again that a countable union of zero-measure sets has measure zero).

Exercise 5

$$f_n : [0,1] \longrightarrow \mathbb{R}, \quad f_n(x) := \begin{cases} e^{x/n}, & x \in [0, 1/n) \end{cases}$$

$$f_n : [0,1] \longrightarrow \mathbb{R} \quad , \quad f_n(x) := \begin{cases} e^{x/n} , & x \in [0, 1/n) \\ 0 & \text{otherwise} \end{cases}$$

Note that if $x = 0$ $f_n(x) = 1$, and if $x \in (0,1]$ then $\exists$ $N(x) > 0$ s.t. $\forall n \geq N(x)$ $f_n(x) = 0$.

Therefore the pointwise limit of f_n is

$$f(x) = \begin{cases} 1 , & x = 0 \\ 0 , & \text{otherwise} \end{cases}$$

This shows that f_n does <u>NOT</u> converge pointwise to zero but converges a.e. to zero (since $\{0\}$ has measure zero).

Moreover, f_n does <u>NOT</u> converge uniformly at zero since

$$\sup_{x \in [0,1]} |f_n(x) - 0| = 1 \underset{n \to \infty}{\not\longrightarrow} 0 .$$

Finally, noting that

$$0 \leq \int_0^1 \underset{\substack{\| \\ e^{x/n}}}{f_n(x)}\, dx \leq \int_0^{1/n} e \, dx \leq \frac{e}{n} \underset{n \to \infty}{\longrightarrow} 0$$

we can conclude that $\lim_{n \to \infty} \int f_n(x)\, dx = 0$ $\square$