

Permuton limits of substitution-closed classes and Baxter permutations: The biased Brownian separable permuton & the (generalized) Baxter permuton

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Substitution-closed classes & the biased Brownian separable permuto

Def.: Let θ be a permutation of size d & $\pi^{(1)}, \dots, \pi^{(d)}$ permutations
The diagram of the permutation $\Theta[\pi^{(1)}, \dots, \pi^{(d)}]$ is obtained by
replacing the i -th dot of θ with the diagram of $\pi^{(i)}$, $\forall i \in [d]$

Example:

$$2413 [132, 21, 1, 12] = \begin{array}{|c|c|c|} \hline & 21 & \\ \hline & & 12 \\ \hline 132 & & \\ \hline & 1 & \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|} \hline & \bullet & & & & & & & & & & \\ \hline & & \bullet & & & & & & & & & \\ \hline & & & \bullet & & & & & & & & \\ \hline & & & & \bullet & & & & & & & \\ \hline & & & & & \bullet & & & & & & \\ \hline & & & & & & \bullet & & & & & \\ \hline & & & & & & & \bullet & & & & \\ \hline & & & & & & & & \bullet & & & \\ \hline & & & & & & & & & \bullet & & \\ \hline & & & & & & & & & & \bullet & \\ \hline \end{array} = 24387156$$

$$\oplus [132, 21] = \begin{array}{|c|c|c|c|c|c|} \hline & & \bullet & & & \\ \hline & \bullet & & & & \\ \hline \bullet & & & & & \\ \hline & & & & & \\ \hline \end{array}$$

$$\ominus [132, 21] = \begin{array}{|c|c|c|c|c|c|} \hline & \bullet & & & & \\ \hline & & \bullet & & & \\ \hline & & & \bullet & & \\ \hline & & & & \bullet & \\ \hline & & & & & \bullet \\ \hline \end{array}$$

Def: A permutation class \mathcal{C} is substitution-closed if for every $\theta, \pi^{(1)}, \dots, \pi^{(10)} \in \mathcal{C}$ then $\theta[\pi^{(1)}, \dots, \pi^{(\theta)}] \in \mathcal{C}$

Def: A permutation is simple if it cannot be written as substitution of smaller permutations ($\sim \frac{n!}{e^2}$ simple perm.)

Fact (Albert-Atkison '05): A class \mathcal{C} is substitution-closed if and only if $\mathcal{C} = \text{Av}(S)$ with S a set of simple perm.

Example: $\mathcal{C} = \text{Av}(2413 - 3142)$ Separable permutations

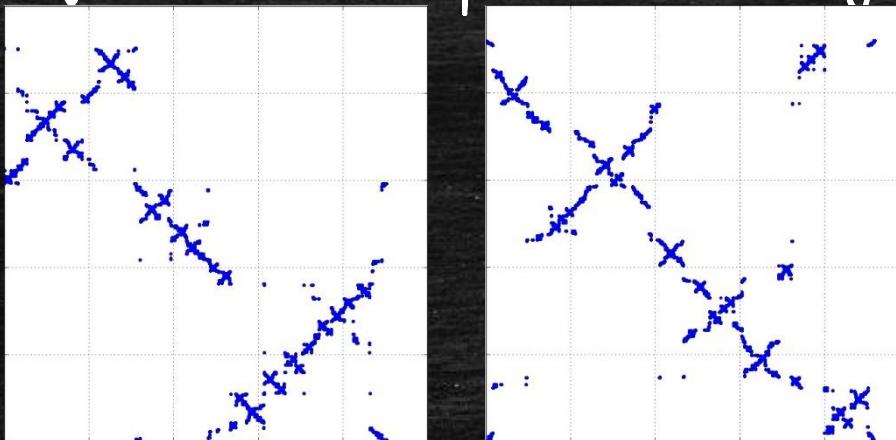
THEOREM [① Bossino, Bouvel, Féray, Gerin, Maazoun and Pierrot, 2016
② B., Bouvel, Féray, Stufler, 2019]

Let \mathcal{C} be a substitution-closed class that satisfies some technical conditions.

Let σ^n be a uniform random permutation of size n in \mathcal{C} . Then

$$\mu_{\sigma^n} \xrightarrow{d} \mu_p =$$

parameter $p \in [0,1]$
depending on \mathcal{C}



RANDOM
BROWNIAN
SEPARABLE
PERMUTON

We start with a general result for permutations

Proposition: [Albert - Atkinson, '05]

A permutation τ of size $n \geq 2$ can be uniquely decomposed as either:

- $\alpha [\pi^{(1)}, \dots, \pi^{(|\alpha|)}]$ where α is simple & $|\alpha| \geq 4$;
- $\oplus [\pi^{(1)}, \dots, \pi^{(d)}]$ where $d \geq 2$ & $\pi^{(i)}$ are \oplus -indecomp.
- $\ominus [\pi^{(1)}, \dots, \pi^{(d)}]$ where $d \geq 2$ & $\pi^{(i)}$ are \ominus -indecomp.

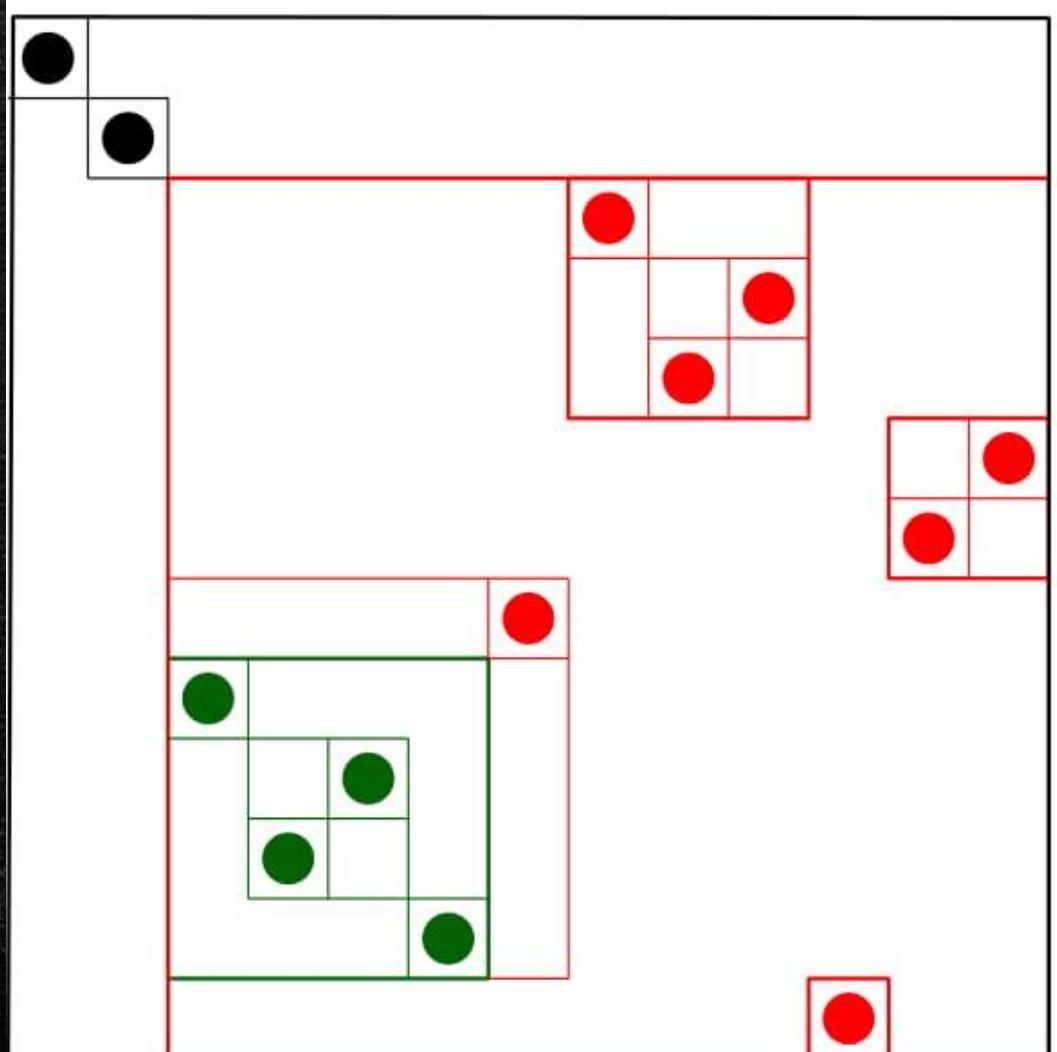
Remark: We can apply the theorem recursively.

Using the theorem recursively, a permutation σ can be naturally encoded by a rooted plane labeled tree, denoted $CT(\sigma)$, as follows:

- If $\sigma = 1$, then $CT(\sigma)$ is reduced to a single leaf.
- If $\sigma = \beta [\pi^{(1)}, \dots, \pi^{(|\beta|)}]$ with $\beta \in \{\text{simple}, \oplus, \ominus\}$ then $CT(\sigma)$ has a root of deg $|\beta|$ labeled by β & the subtrees attached to the root are

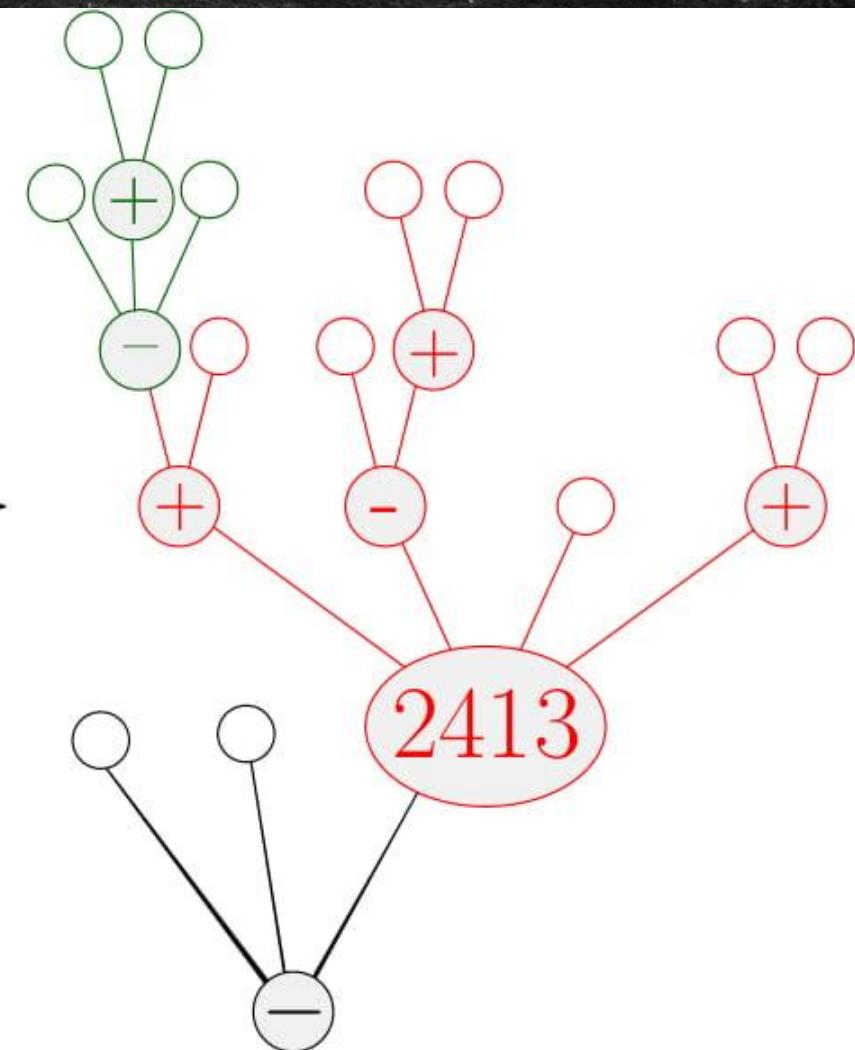
$$CT(\pi^{(1)}), \dots, CT(\pi^{(|\beta|)})$$

in this order from left to right.



$$\sigma = 1 \ 2 \ 7 \ 5 \ 6 \ 4 \ 8 \ 13 \ 11 \ 12 \ 3 \ 9 \ 10$$

CT →



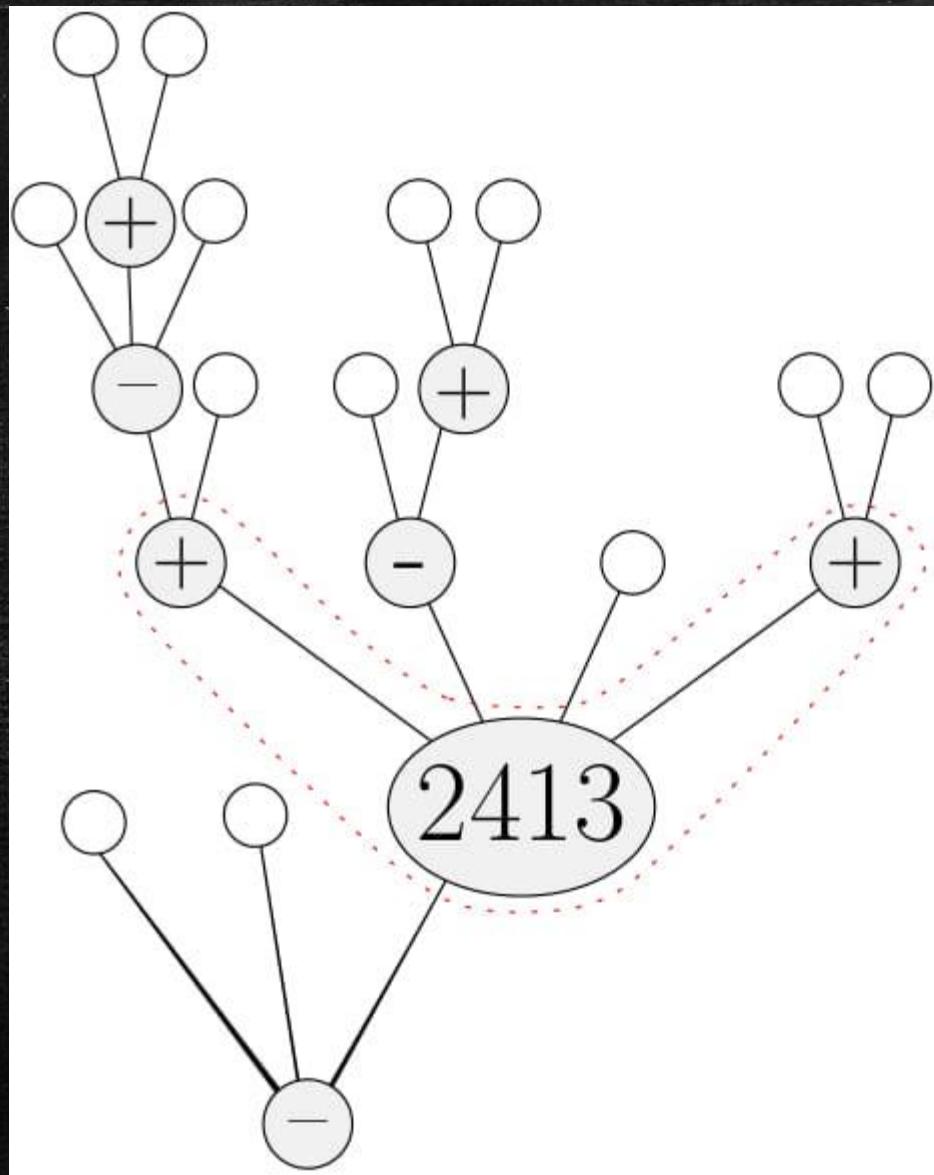
Proposition: Let π be a uniform random permutation of size n in \mathcal{C} and define $\pi^{(1)}, \dots, \pi^{(d)}$ as before.

Let m be an index such that

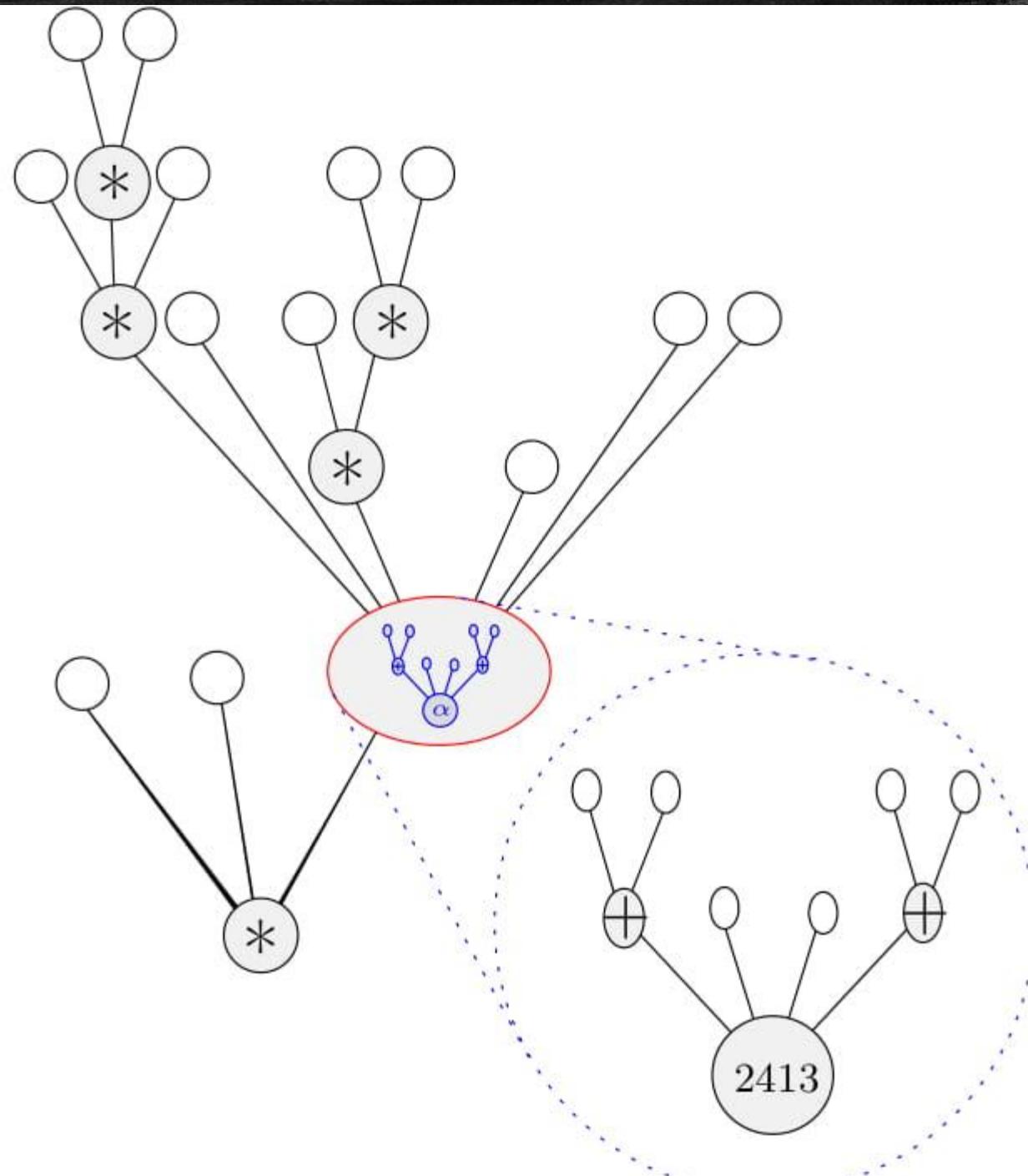
$$|\pi^{(m)}| := \max(|\pi^{(1)}|, \dots, |\pi^{(d)}|)$$

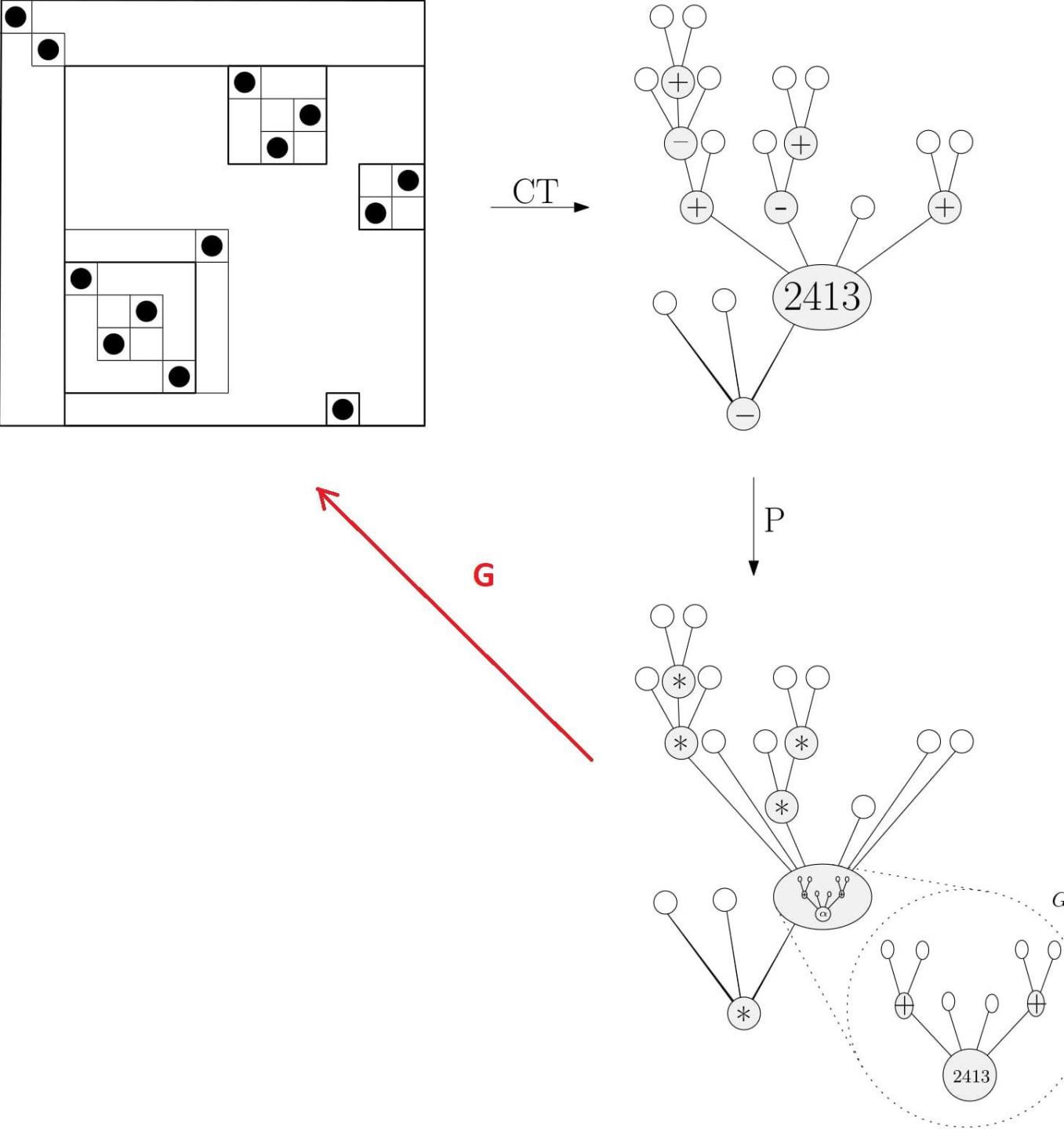
Then $\Delta := n - |\pi^{(m)}|$ is stochastically bounded & conditionally on m and Δ , $\pi^{(m)}$ is a uniform random \oplus -indecomposable perm of size $n - \Delta$ in \mathcal{C} .

QUESTION: How do we sample a uniform random \oplus -indecomposable permutation in \mathcal{C} of size n ?



P →





QUESTION: How do we sample a uniform random \oplus -indecomposable permutation in \mathcal{C} of size n ?

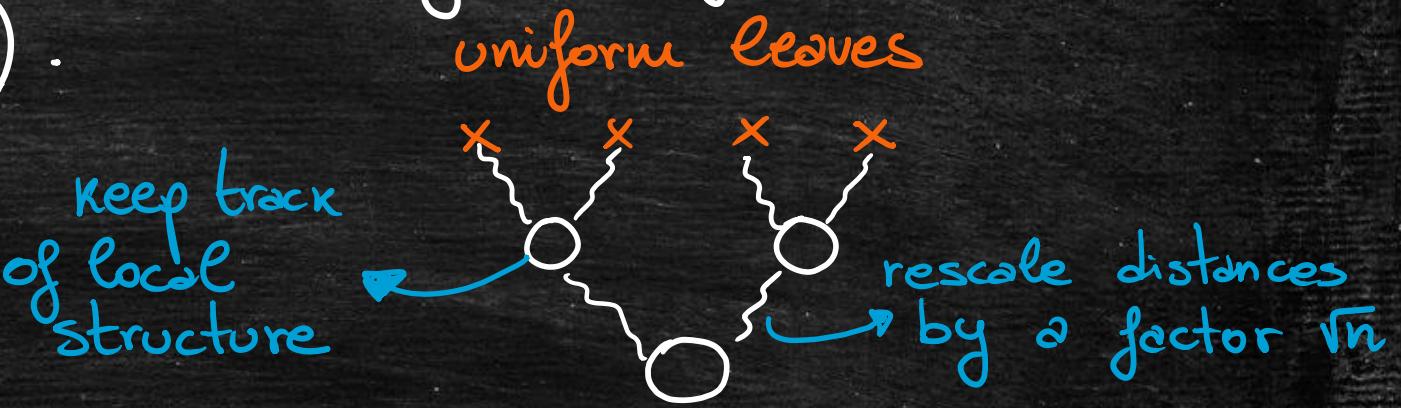
Answer: We sample a uniform packed tree with n -nodes.

- We consider a GW-tree T^ε where ε is a specific reproduction law depending on the generating series of the class \mathcal{C} .
- We decorate each internal vertex v of T^ε with a uniformly at random $d_{T^\varepsilon}^+(v)$ -sized admissible decoration for packed trees, denoted $\lambda_{T^\varepsilon}(v)$.

Proposition: $(T^\varepsilon, \lambda_{T^\varepsilon} | |T^\varepsilon|_p = n)$ is a unif. packed tree with n leaves.

Rest of the proof:

- Theorem for the semi-local convergence of the skeleton of $(T^\varepsilon, \lambda_{T^\varepsilon} \mid |T^\varepsilon|_e = n)$.

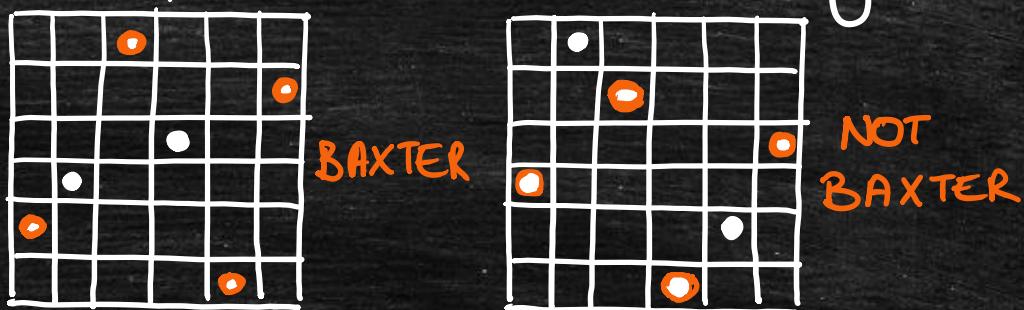


- Transfer the result to permutations using G & the theorem that we saw in Lecture 1.

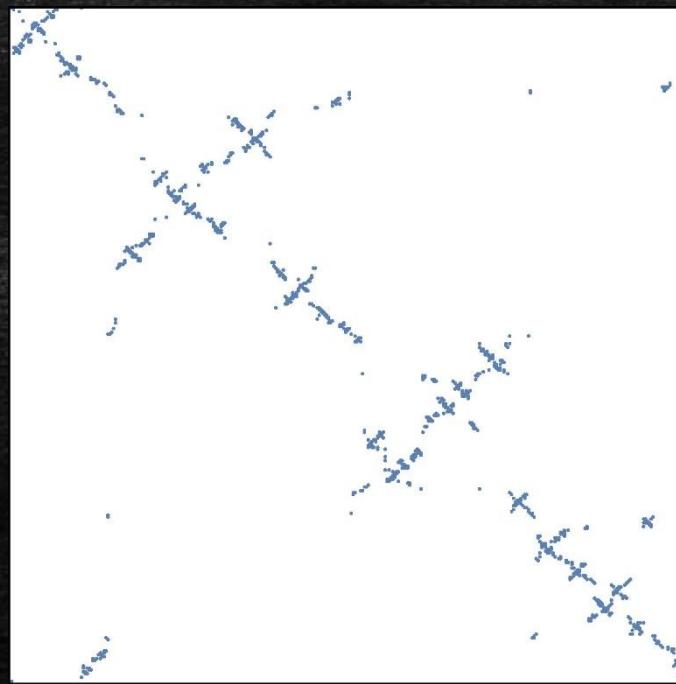
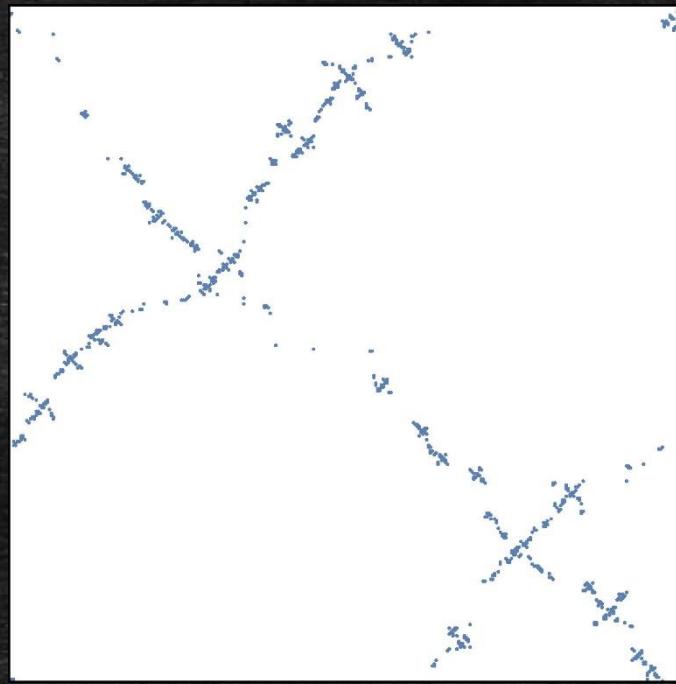
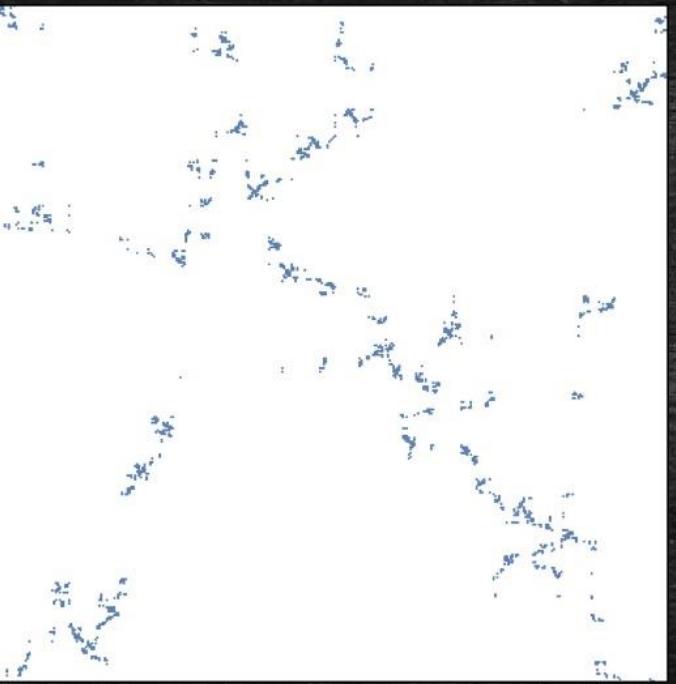
Baxter permutations & the (generalized) Baxter permton

Def: Baxter permutations are permutations avoiding the patterns

$2\boxed{4}13$ & $31\boxed{4}2$.



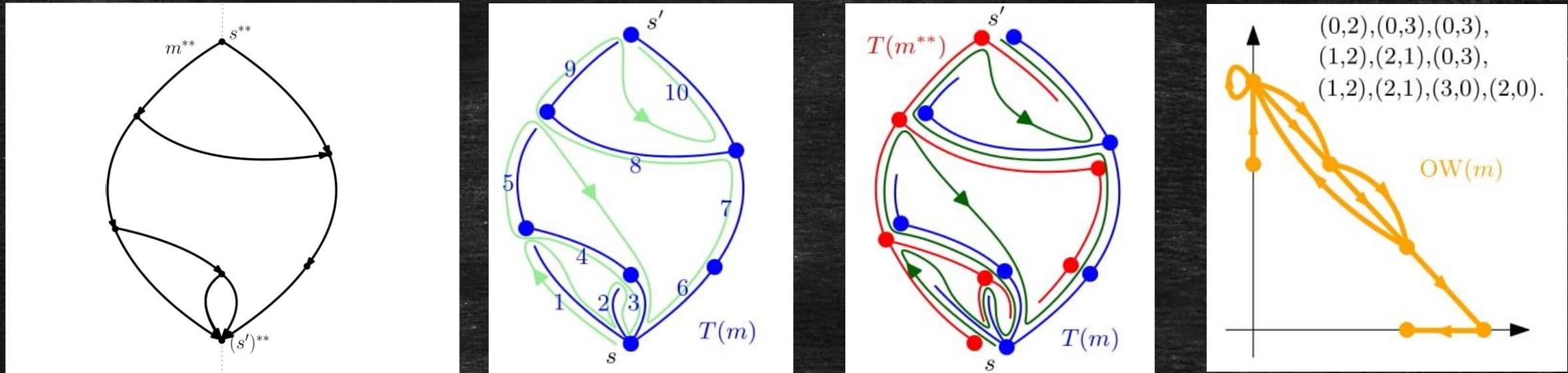
Dokos & Pak (2014) explored the expected shape of doubly alternating Baxter permutations, i.e. Baxter perm. σ s.t. σ and σ^- are alternating and they claimed that
"IT WOULD BE NICE TO COMPUTE THE LIMIT SHAPE OF BAXTER PERMUTATIONS"



Motivations(PART 2): Bipolar orientations and walks in cones

Bonichon, Bousquet-Mélou & Fusy (2011) showed that Baxter permutations are in bijection with plane bipolar orientations.

Def: A PLANE BIPOLAR ORIENTATION is a planar map (connected graphs properly embedded in the plane up to continuous deformations) equipped with an acyclic orientation of the edges with exactly one source (a vertex with only outgoing edges) and one sink (a vertex with only incoming edges) both on the outer face.



Kenyon, Miller, Sheffield & Wilson (2015) constructed the following bijection.

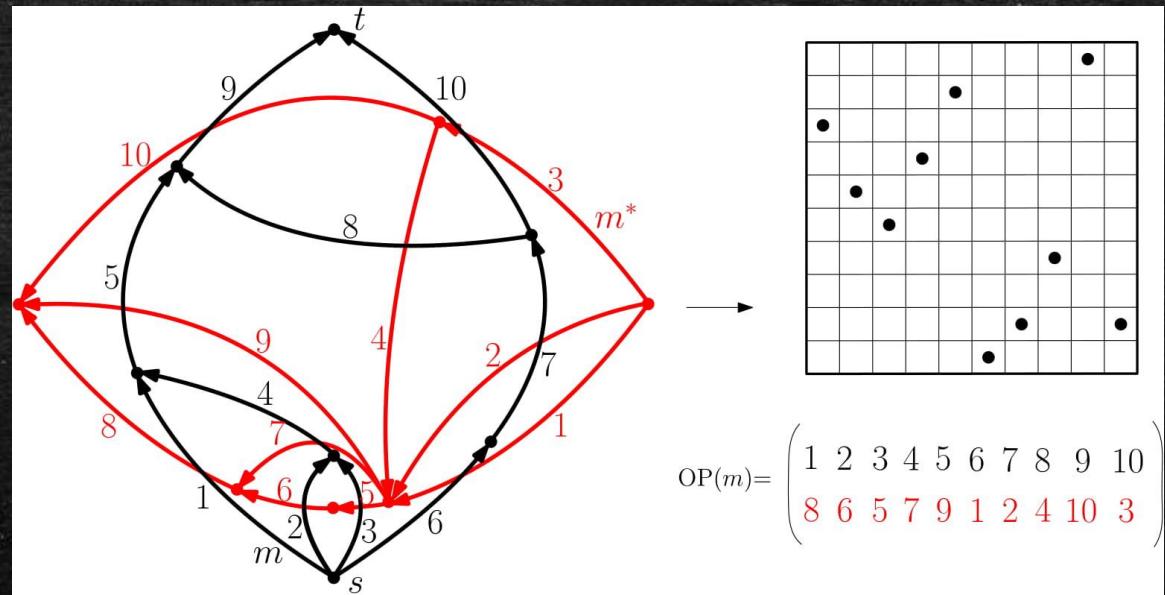
Def: Let $n \geq 1$ and m be a bipolar orientation with n edges. We define

$OW(m) = (X_t, Y_t)_{1 \leq t \leq n} \in (\mathbb{Z}_{\geq 0}^2)^n$ as follows: for $1 \leq t \leq n$, X_t is the height in the tree $T(m)$ of the bottom vertex of e_t and Y_t is the height in the tree $T(m^{**})$ of the top vertex of e_t .

THEOREM: (Gwynne, Holden, Sun 2016)

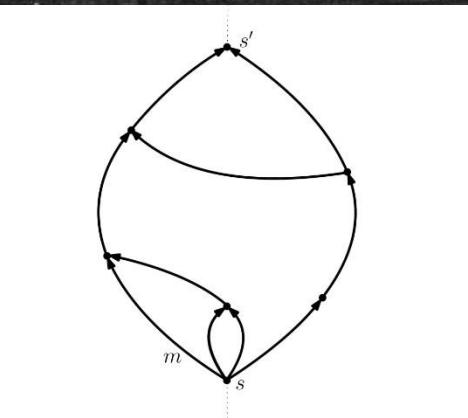
The pairs of height functions for an infinite-volume random bipolar triangulation and its dual converge jointly in law to the two Brownian motions which encode the same $\sqrt{4/3}$ -LQG surface decorated by both an SLE_{12} and the "dual" SLE_{12} which travels in a perpendicular direction.

Def: A TANDEM WALK is a two-dimensional walk in $\mathbb{Z}_{\geq 0}^2$ starting at $(0,0)$ and ending at $(k,0)$ with steps in $\{(+1,-1)\} \cup \{(-i,j) : i, j \geq 0\}$.

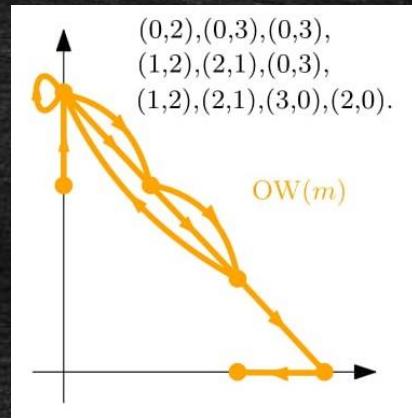


Def: Let $n \geq 1$ and m a bipolar orientation with n edges.
 Let $OP(m)$ be the only permutation π such that for every $1 \leq i \leq n$, the i -th edge to be visited in the exploration of $T(m)$ corresponds to the $\pi(i)$ -th edge to be visited in the exploration of $T(m^*)$.

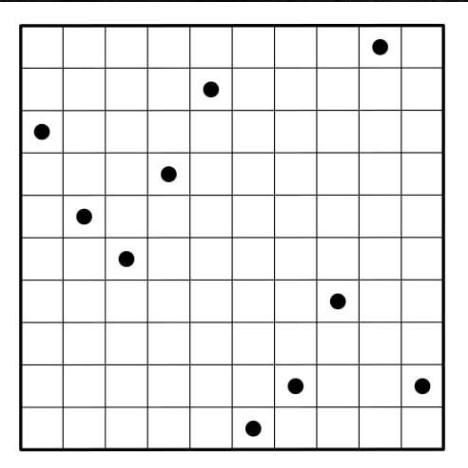
So FAR ...



$\xrightarrow{\text{OW}}$



$\downarrow \text{OP}$



$\xrightarrow{\text{OP} \circ \text{OW}^{-1}}$



WE WANT "TO READ"
THE PATTERNS OF A
PERMUTATION
IN THE CORRESPONDING
WALK

Coalescent-walk processes

Let $W_t = (X_t, Y_t)$ be a tandem walk & $\sigma = \text{OP} \circ \text{OW}^{-1}(W)$ be the corresponding Baxter permutation.

IDEA: Given $i < j$, we want to find a way in order to "read" in W_t if $\sigma(i) < \sigma(j)$ or $\sigma(j) < \sigma(i)$.

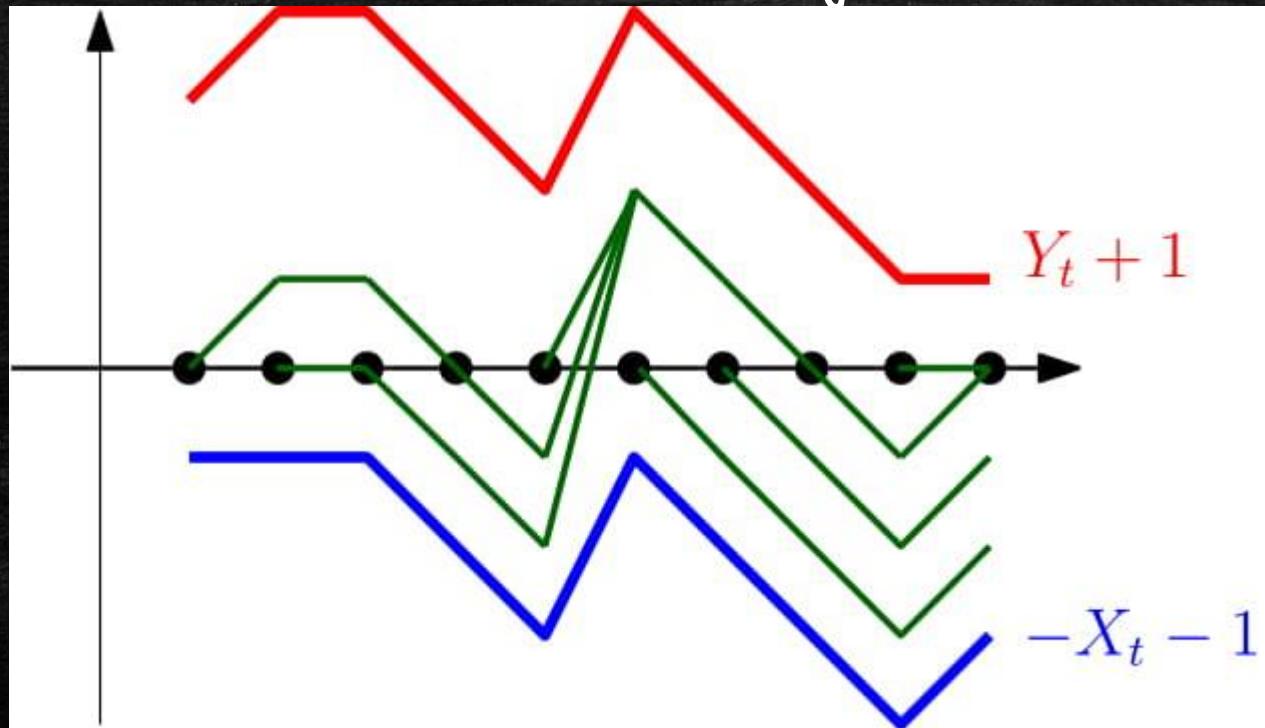
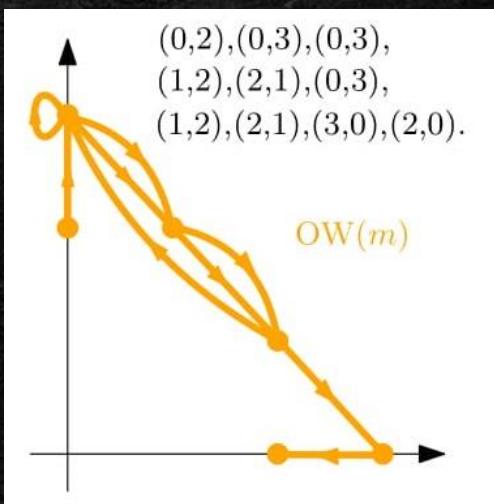
SOLUTION: COALESCENT - WALK PROCESSES

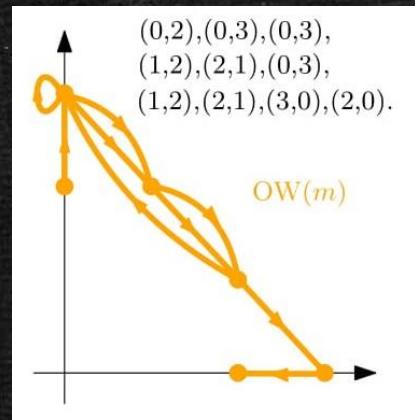
i.e. a collection of walks $(Z_i^{(t)})_t$ that "follow" Y_t when they are positive and $-X_t$ when they are negative.

Def: Let $(W_t)_{t \in [n]} = (X_t, Y_t)_{t \in [n]}$ be a tandem walk of length $n \in \mathbb{N}$.
 The COALESCENT-WALK PROCESS associated to $(W_t)_{t \in [n]}$ is a collection of n one-dimensional walks $(Z^{(t)})_{t \in [n]} =: WC(W)$ defined for every $t \in [n]$ by:

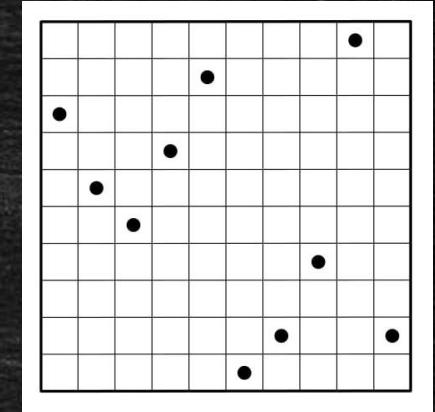
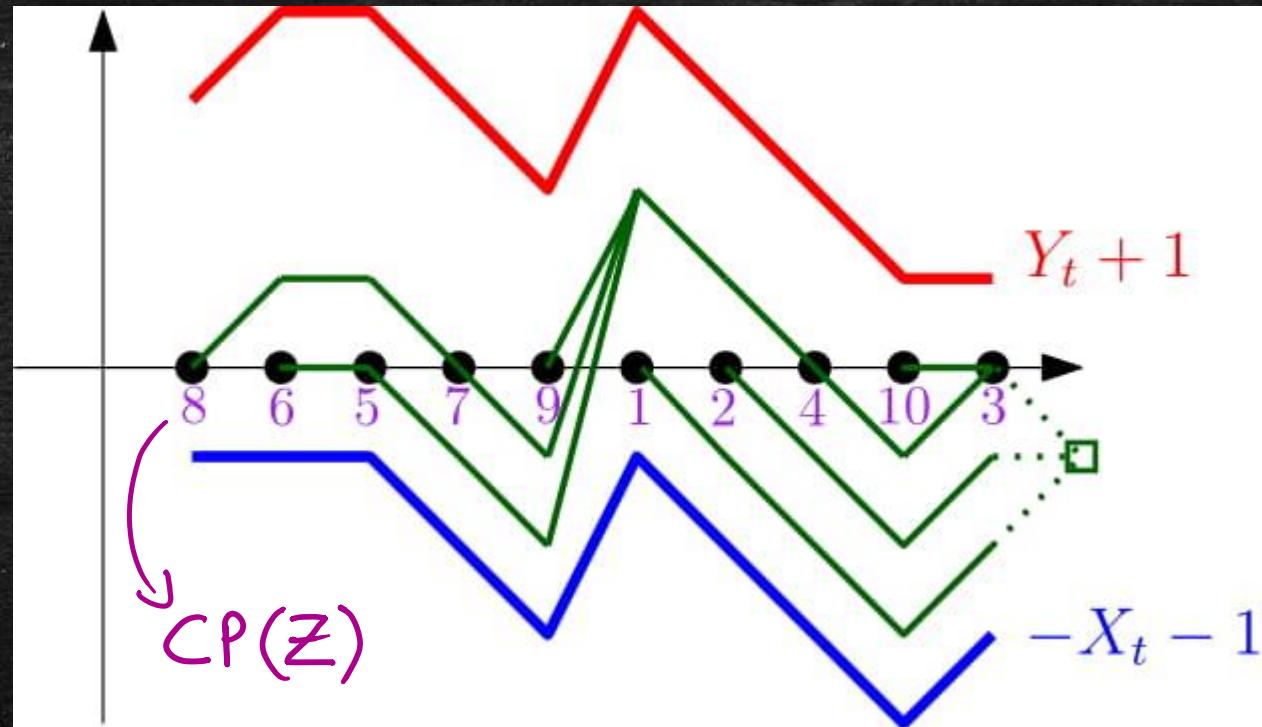
$$\bullet Z_t^{(t)} = 0$$

$$\bullet Z_k^{(t)} = \begin{cases} Z_{k-1}^{(t)} + (Y_k - Y_{k-1}) & \text{if } Z_{k-1}^{(t)} > 0 \\ Z_{k-1}^{(t)} - (X_k - X_{k-1}) & \text{if } Z_{k-1}^{(t)} < 0 \text{ and } Z_{k-1}^{(t)} - (X_k - X_{k-1}) < 0 \\ Y_k - Y_{k-1} & \text{if } Z_{k-1}^{(t)} < 0 \text{ and } Z_{k-1}^{(t)} - (X_k - X_{k-1}) \geq 0 \end{cases}$$





$$W = (W_t)_{t \in [n]}$$

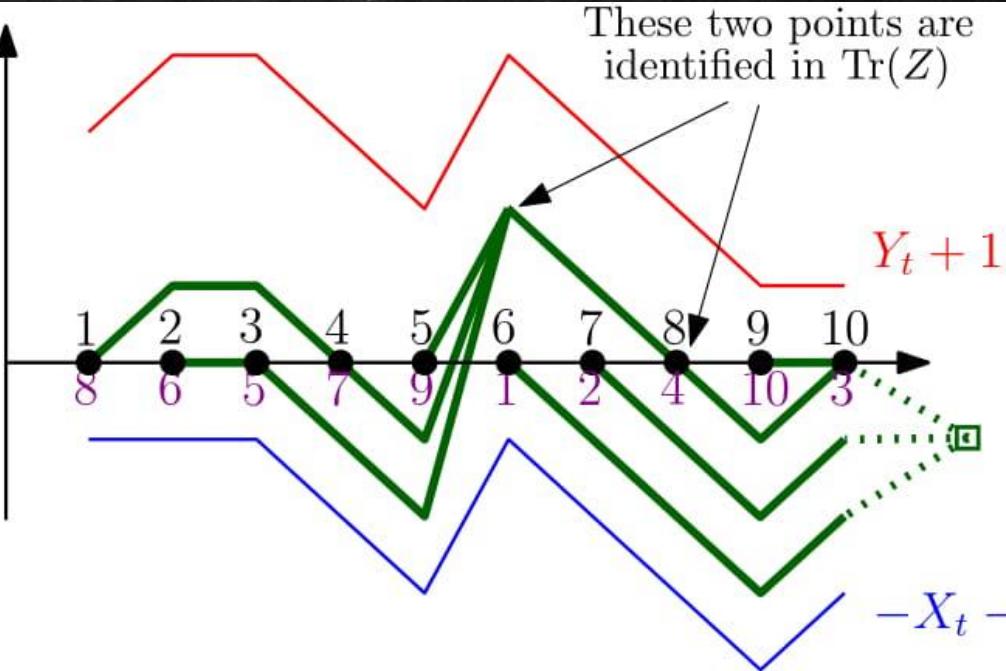
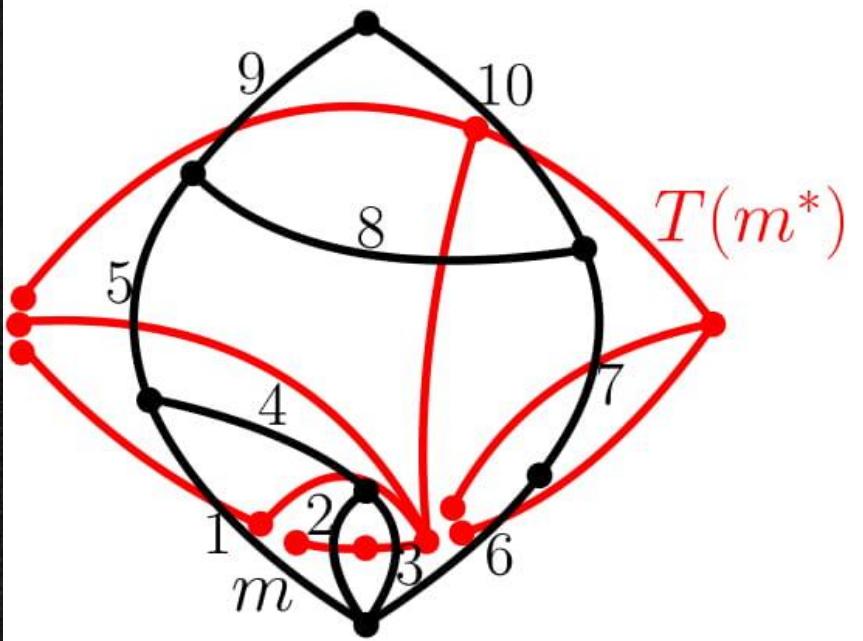


$$\sigma = OP \circ OW^{-1}(w)$$

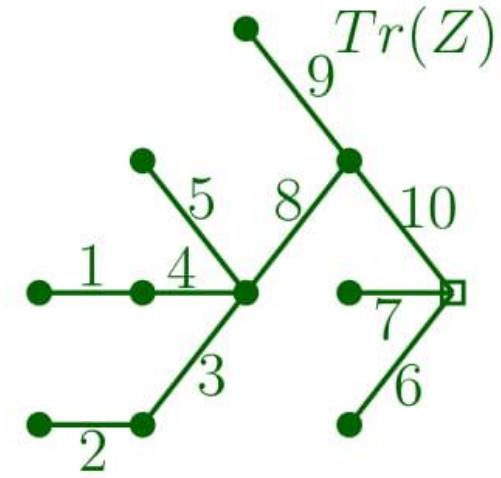
THEOREM: Let $W = (W_t)_{t \in [n]}$ be a tandem walk and $\sigma = OP \circ OW^{-1}(w)$ the corresponding Baxter permutation. Then

$$\underbrace{CP \circ WC(W)}_{\text{coalescent-walk process } Z} = \sigma.$$

coalescent-walk process Z



$CP(Z) = 8 \ 6 \ 5 \ 7 \ 9 \ 1 \ 2 \ 4 \ 10 \ 3$



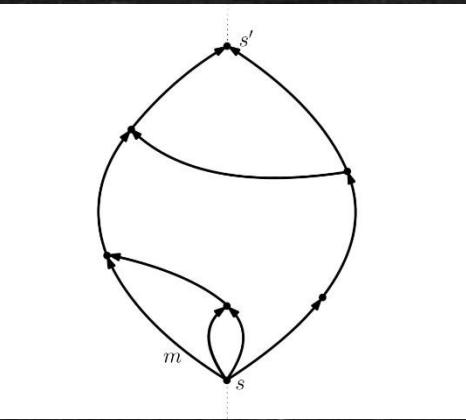
PROPOSITION: Let σ be a Baxter permutation of size $n \in \mathbb{N}$ corresponding to a coalescent-walk process $(Z^{(t)})_{t \in [n]}$. Then for $i < j$

$$\sigma(i) < \sigma(j) \Leftrightarrow Z_j^{(i)} < 0$$

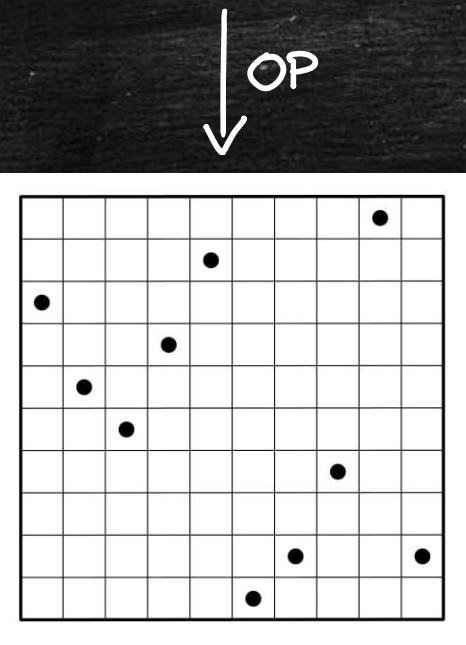
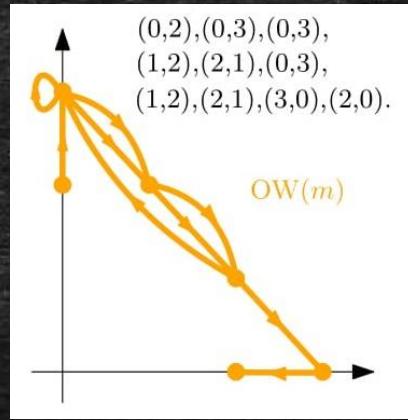
THEOREM: Let m be a bipolar orientation with n edges and $Z = (Z^{(t)})_{t \in [n]}$ be the corresponding coalescent-walk process. Then

$$Tr(Z) = T(m^*) \text{ as labeled trees.}$$

THEOREM :
 (B.-Maazoun)
 2020

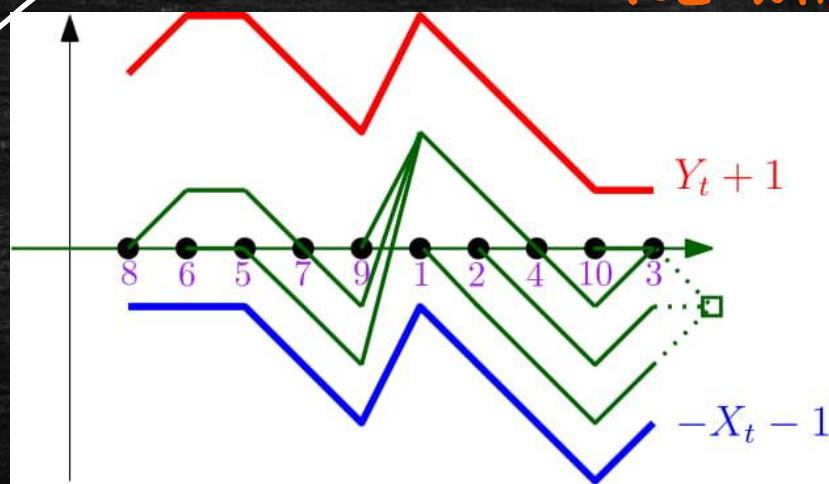


OW



CP

WC

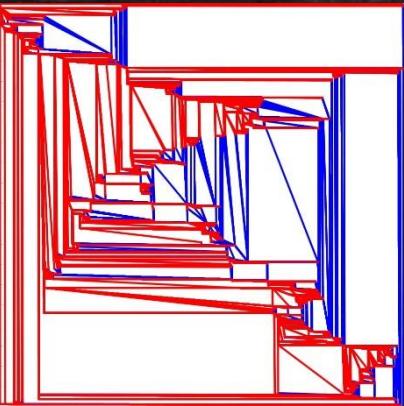


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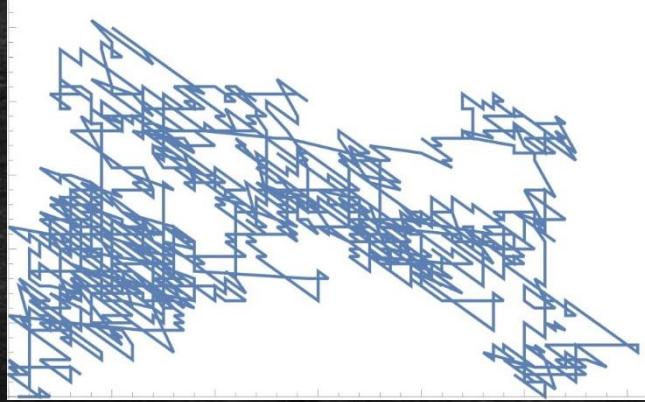


This diagram commutes.

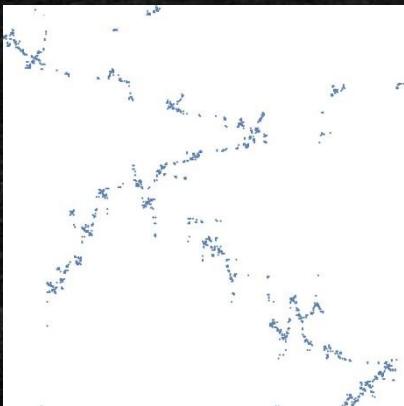
THEOREM :
(B.-Maazoun)
2020



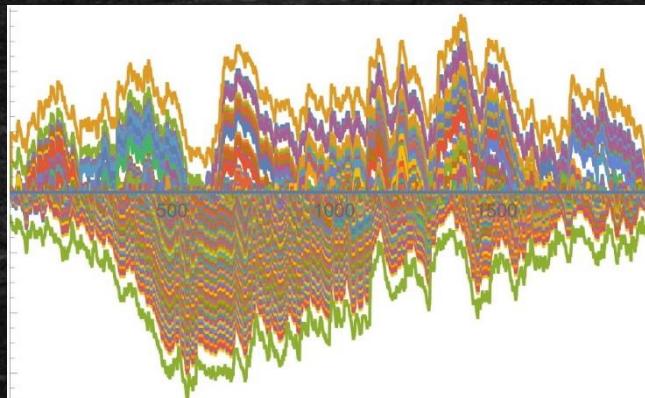
$\xrightarrow{\text{OW}}$



$\downarrow \text{OP}$

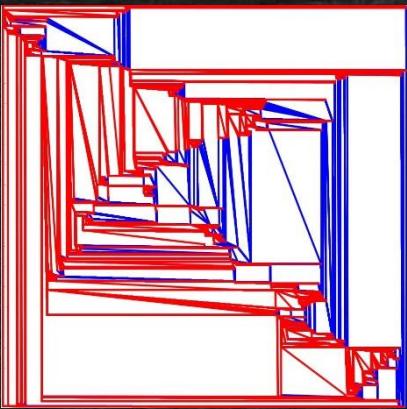


$\downarrow \text{WC}$

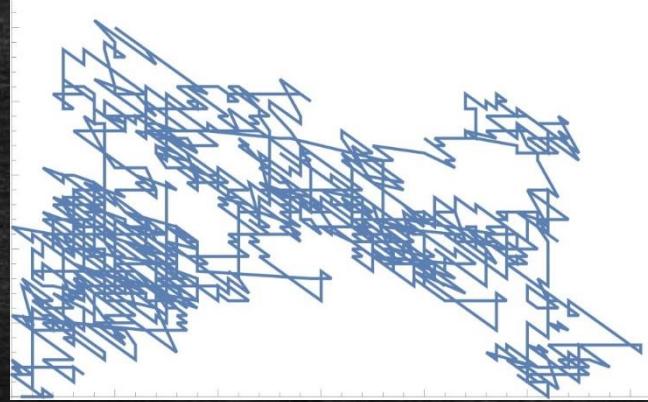


$\xleftarrow{\text{CP}}$

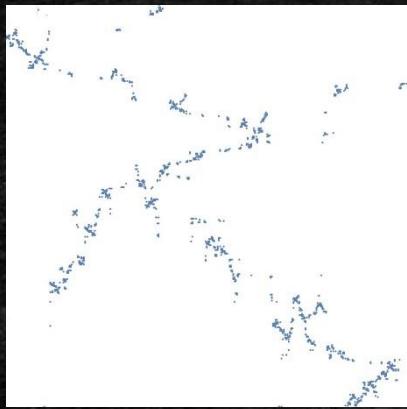
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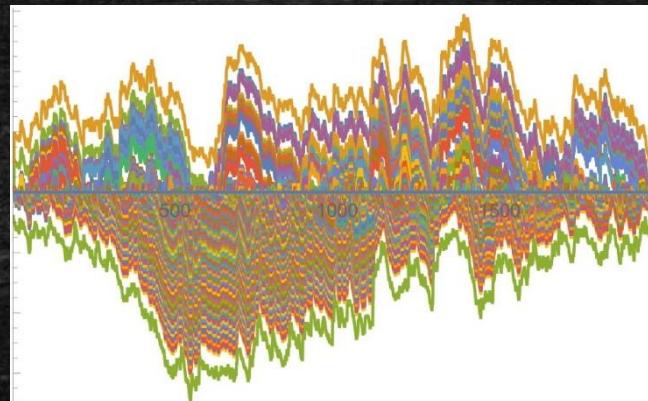
\xrightarrow{OW}



$\downarrow OP$



$\downarrow WC$



\xleftarrow{CP}

Scaling limits of coalescent-walk processes

The continuous coalescent-walk process

Consider a two dimensional process $W(t) = (X(t), Y(t))_{t \in I}$ and the following family of stochastic differential equations (SDEs) indexed by $\mu \in I$

$$(*) \quad \begin{cases} dZ^{(\mu)}(t) = \mathbb{1}_{\{Z^{(\mu)}(t) > 0\}} dY(t) - \mathbb{1}_{\{Z^{(\mu)}(t) \leq 0\}} dX(t), & t \in (\mu, \infty) \cap I, \\ Z^{(\mu)}(t) = 0, & t \in (-\infty, \mu] \cap I. \end{cases}$$

THEOREM (Prokaj 2013, Çağlar - Hajri - Kardakos 2018)

Let $(W(t))_{t \in I}$ be a two-dimensional Brownian motion with covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ for some $\rho \in (-1, 1)$. Fix $\mu \in I$. We have path-wise uniqueness and existence of a strong solution for the SDE (*) driven by $W(t)$.

We now consider the SDEs (*) driven by a two-dimensional Brownian excursion $W_e = (X_e, Y_e)$ with cov. matrix $\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$:

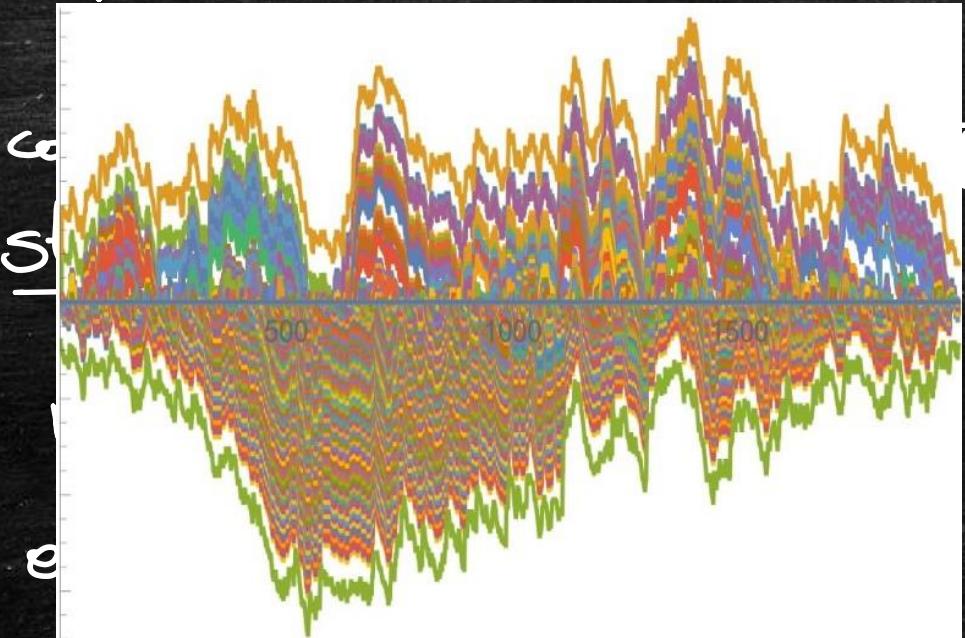
$$\begin{cases} dZ_e^{(u)}(t) = \mathbb{1}_{\{Z_e^{(u)}(t) > 0\}} dY_e(t) - \mathbb{1}_{\{Z_e^{(u)}(t) \leq 0\}} dX_e(t), & t > u, \\ Z_e^{(u)}(t) = 0, & t \leq u. \end{cases}, \quad \forall u \in [0,1]$$

Using absolutely uniqueness of



The above

For almost all e



proved existence &

imply that:

for almost every $u \in [0,1]$.

Def: We call CONTINUOUS COALESCENT-WALK PROCESS (driven by W_e) the collection of solutions $\{Z_e^{(u)}\}_{u \in [0,1]}$ where properly defined.

Let $\bar{W} = (\bar{X}, \bar{Y}) = (\bar{X}_k, \bar{Y}_k)_{k \geq 0}$ be a two dimensional random walk having value $(0,0)$ at time 0 and step distribution

$$Y = \frac{1}{2} \delta_{(+1, -1)} + \sum_{i,j \geq 0} 2^{-i-j-3} \delta_{(-i, j)}$$

Proposition: The following is a uniform tandem walk of length n :

$$(W_t)_{1 \leq t \leq n} := \left((\bar{W}_t)_{1 \leq t \leq n} \mid \bar{W}_0 = (0,0), \bar{W}_{n+1} = (0,0), (\bar{W}_t)_{0 \leq t \leq n+1} \in (\mathbb{Z}_{\geq 0}^2)^{n+1} \right)$$

Let $Z = WC(W)$ be the corresponding coalescent-walk process and $(L^{(i)}(j))_{-\infty < i < j < \infty}$ be the corresponding local time process. We define the rescaled continuous versions: for all $n \geq 1, u \in \mathbb{R}$, let

$$W_n : \mathbb{R} \rightarrow \mathbb{R}^2 \quad Z_n^{(u)} : \mathbb{R} \rightarrow \mathbb{R} \quad L_n^{(u)} : \mathbb{R} \rightarrow \mathbb{R}$$

be the continuous functions defined by interpolating the following points:

$$W_n\left(\frac{k}{n}\right) = \frac{1}{\sqrt{2n}} W_k \quad Z_n^{(u)}\left(\frac{k}{n}\right) = \frac{1}{\sqrt{2n}} Z_K^{(\lceil nu \rceil)} \quad L_n^{(u)}\left(\frac{k}{n}\right) = \frac{1}{\sqrt{2n}} L^{(\lceil nu \rceil)}(k), \quad k \in \mathbb{R}$$

THEOREM: Let $\omega \in (0,1)$. We have the following joint convergence in $C([0,1], \mathbb{R})^3 \times C([0,1], \mathbb{R})$

(B.-Maazoun)
2020

$$(W_n, Z_n^{(\omega)}, L_n^{(\omega)}) \xrightarrow[n \rightarrow \infty]{d} (W_e, Z_e^{(\omega)}, L_e^{(\omega)})$$

↙ 2-dim. Brownian excursion
 ↓ associated
 in the quadrant with cov $\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$ continuous
 ↘ associated local time
 local-walk process

- Remarks**:
- The convergence to the process W_e is due to Denisov & Wachtel
 - The convergence of local times is up to time 1 excluded!

THEOREM: Let $(\epsilon_i)_{i \geq 1}$ be a sequence of iid uniform random variables

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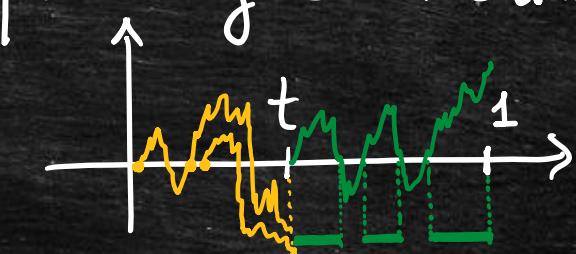
on $[0,1]$ independent of all other variables. Then

$$(W_n, (Z_n^{(\epsilon_i)}, L_n^{(\epsilon_i)}))_{i \geq 1} \xrightarrow{d} (W_e, (Z_e^{(\epsilon_i)}, L_e^{(\epsilon_i)}))_{i \geq 1}$$

Scaling limits of Baxter permutations

Let $\mathcal{Z}_e = \{\mathcal{Z}_e^{(u)}\}_{u \in (0,1)}$ be the family of solutions of the SDEs (*) driven by the Brownian excursion W_e in the quadrant of cov. matrix $\begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & -\frac{1}{2} \end{pmatrix}$.

Define the following random function for $t \in [0,1]$:



$$\varphi_{\mathcal{Z}_e}(t) := \text{Leb}\left(\{x \in [0,t] \mid \mathcal{Z}_e^{(x)}(t) < 0\}\right) + \text{Leb}\left(\{x \in [t,1] \mid \mathcal{Z}_e^{(t-x)}(x) > 0\}\right)$$

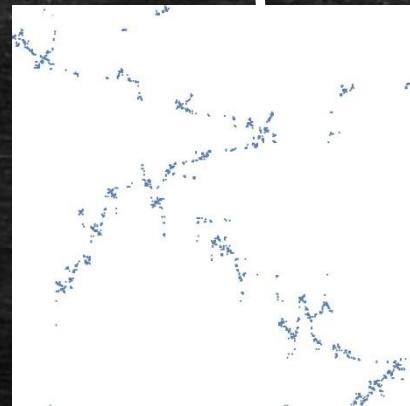
The BAXTER PERMUTON is the following random probability measure on the square $[0,1]^2$:

$$\mu_{\mathcal{Z}_e}(\cdot) := (\text{Id}, \varphi_{\mathcal{Z}_e})_* \text{Leb}(\cdot) = \text{Leb}(\{t \in [0,1] \mid (t, \varphi(t)) \in \cdot\}).$$

PROPOSITION: $\mu_{\mathcal{Z}_e}(\cdot)$ is almost surely a permutoon.

THEOREM: Let σ_n be a uniform Baxter permutation of size n .
(B.-Massoum 2020) We have the following convergence in the space of permutoons

$$\mu_{\sigma_n} \xrightarrow{d} \mu_{\mathcal{Z}_e}.$$



Proof based on:

• PROPOSITION: $\sigma(i) < \sigma(j) \Leftrightarrow Z_j^{(i)} < 0$

• THEOREM: $(W_n, (Z_n^{(u_i)}, L_n^{(u_i)})_{i \geq 1}) \xrightarrow{d} (W_e, (\mathcal{Z}_e^{(u_i)}, \mathcal{L}_e^{(u_i)})_{i \geq 1})$

Future projects

Fix $\rho \in [-1, 1]$ and $q \in [0, 1]$. Consider a two-dim. Brow. excursion

$\mathcal{E}_\rho = (X_\rho, Y_\rho)$ with cov. matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ & the SDEs

$$(*) \begin{cases} dZ_{\rho, q}^{(u)}(t) = \mathbb{1}_{\{Z_{\rho, q}^{(u)}(t) > 0\}} dY_\rho(t) - \mathbb{1}_{\{Z_{\rho, q}^{(u)}(t) \leq 0\}} dX_\rho(t) + (Z_q - 1) d\mathcal{L}^Z(t), & t > u \\ Z_{\rho, q}(t) = 0, & t \leq u \end{cases},$$

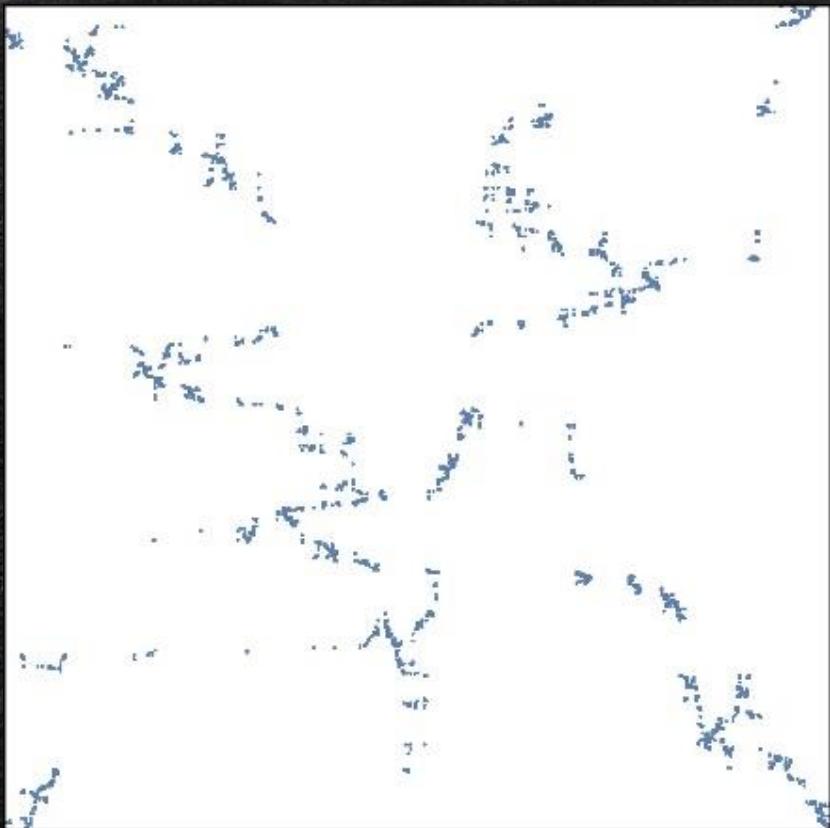
where $\mathcal{L}^Z(t)$ is the local time process of $Z_{\rho, q}^{(u)}(t)$ at zero.

From $(*)$ we can define $\mu_{\rho, q} := \mu_{Z_{\rho, q}}$. BAXTER PERMUTON = $\mu_{-1/2, 1/2}$

CONJECTURE: The Brownian sep. permutoon μ_ρ satisfies

$$\mu_\rho \stackrel{d}{=} \mu_{-1, 1-\rho}.$$

The permutoon $\mu_{\rho, q}$ is a NEW UNIVERSAL LIMITING OBJECT.



Baxter

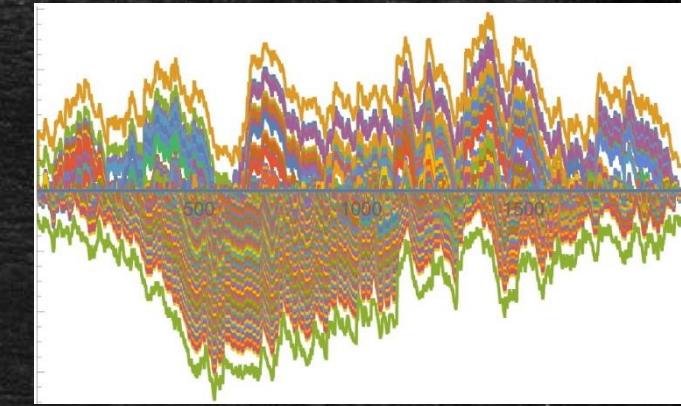
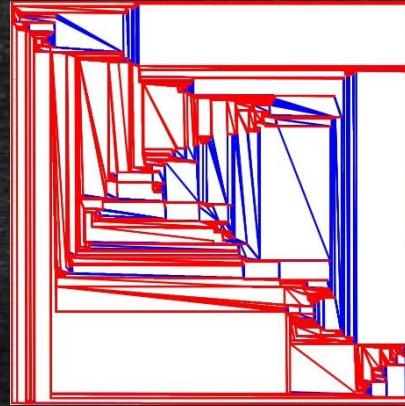
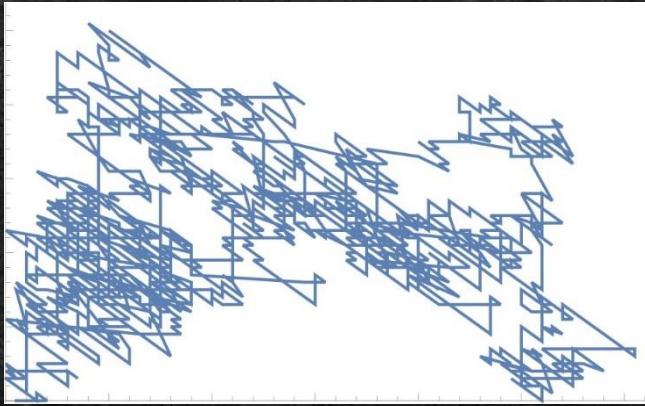
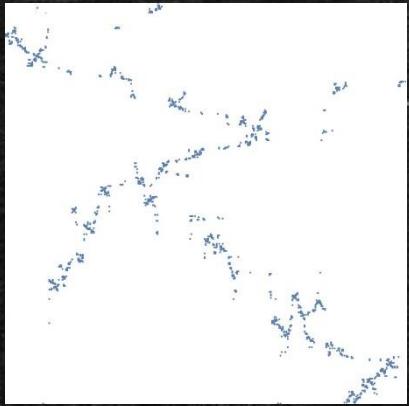


$$\mathcal{M}_{-\frac{1}{2}, \frac{1}{2}}$$

CONJ: Semi-Baxter



$$\mu_{f,q} \quad \text{with} \quad |f| < 1 \\ q \neq \frac{1}{2}$$



THANK YOU!