

Lecture 1

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The limiting shape of random permutations: An introduction to permuton convergence

Plan of the course:

- LECTURE 1: Introduce a notion of convergence for permutations
 - good notion of scaling limit
(like Gromov-Hausdorff top.)
 - convergence of substructures
(like graphons)
- LECTURE 2: Universality of permutation limits + connections with other well and less-well known objects.

1. Definitions & notation: pattern-avoiding permutations

Def: A permutation of size $n \in \mathbb{Z}_{>0}$ is a bijection from the set $[n] = \{1, \dots, n\}$ to itself. We often use the one-line notation to denote a permutation, that is

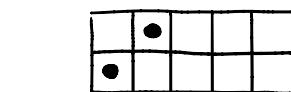
$$\sigma = \sigma(1) \sigma(2) \dots \sigma(n).$$

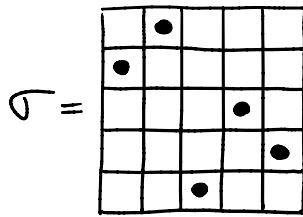
The size of a permutation σ is denoted by $|\sigma|$.

We denote by S_n the set of permutations of size n and by S the set of permutations of finite size.

Def: The diagram of a permutation σ of size n is a $n \times n$ cartesian diagram with n dots at position $(i, \sigma(i))_{i \in [n]}$.

Example: If $\sigma = 4 5 1 3 2$ then the diagram of σ is





Notation: If x_1, \dots, x_n is a sequence of distinct numbers, let $\text{std}(x_1, \dots, x_n)$ be the unique permutation $\pi \in S_n$ such that $\pi(i) < \pi(j)$ iff $x_i < x_j$. Given a permutation σ and an interval $I \subseteq [|\sigma|]$, let $\text{pat}_I(\sigma) := \text{std}((\sigma_i)_{i \in I})$.

Example: If $\sigma = 45132$ & $I = \{1, 3, 5\}$ then

$$\text{pat}_I(\sigma) = \text{std}(4, 1, 2) = 312.$$

Def: Given two permutations $\sigma \in S_n$, $\pi \in S_k$ with $n \geq k$ we say that σ contains π as a pattern if there exists a subset $I \subseteq [n]$ such that $\text{pat}_I(\sigma) = \pi$. If $I = \{i_1, \dots, i_k\}$ then $\sigma(i_1) \dots \sigma(i_k)$ is called an occurrence of the pattern π in σ . We denote by $\text{occ}(\pi, \sigma)$ the number of occurrences of a pattern π in σ , more precisely

$$\text{occ}(\pi, \sigma) = \#\{I \subseteq [n] \mid \text{pat}_I(\sigma) = \pi\}.$$

Moreover, we denote by $\widetilde{\text{occ}}(\pi, \sigma)$ the proportion of occurrences of a pattern π in σ , that is

$$\widetilde{\text{occ}}(\pi, \sigma) = \frac{\text{occ}(\pi, \sigma)}{\binom{n}{k}} \in [0, 1].$$

Def: A permutation σ avoids a pattern π if $\text{occ}(\pi, \sigma) = 0$.

Given a set of patterns $B \subseteq S$, we denote by $\text{Av}_n(B)$ the set of permutations of size n that avoid all the patterns in B

Given a set of patterns $\omega = \omega$, we denote by $\text{Av}(\omega)$ the set of permutations of size n that avoid all the patterns in ω and by $\text{Av}(\mathcal{B})$ the set $\bigcup_{n \in \mathbb{N}_{\geq 0}} \text{Av}_n(\mathcal{B})$.

Example: The permutation $\sigma = 45132$ avoids $\pi = 123$ since it is not possible to find 3 elements of σ that are in increasing order.

2. Permutons

A nice reference for this section is Section 2 of "Universal limits of substitution-closed permutation classes" by Bassino, Bouvel, Féray, Gerin, Maazoun, Pierron.

2.1 Deterministic permutons

Def: A permton μ is a probability measure on the unit square $[0,1]^2$ with uniform marginals, i.e. $\forall a < b \in [0,1]$

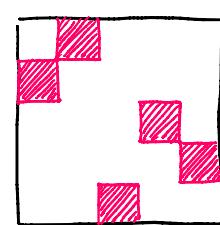
$$\mu([a,b] \times [0,1]) = \mu([0,1] \times [a,b]) = b-a.$$

Example: The two-dimensional Lebesgue measure on $[0,1]^2$ is a permton.

Given a permutation $\sigma \in S_n$ there is a "natural way" to identify it with a permton μ_σ defined by:

$$\mu_\sigma(A) = n \sum_{i=1}^n \text{Leb}\left([\frac{i-1}{n}, \frac{i}{n}] \times [\frac{\sigma(i)-1}{n}, \frac{\sigma(i)}{n}] \cap A\right), \quad \forall A \in \overbrace{\mathcal{B}([0,1]^2)}^{\text{Borel sets of } [0,1]^2}$$

Example: If $\sigma = 45132$ then $\mu_\sigma =$

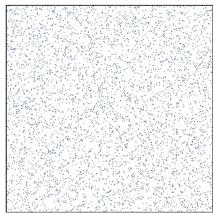


↳ Lebesgue measure of total mass $\frac{1}{6}$

Lebesgue measure
of total mass $\frac{1}{n}$

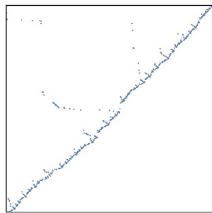
Intermezzo: Some nice pictures

Lebesgue measure



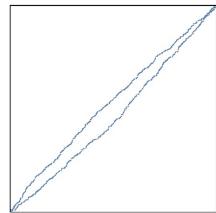
S

Hoffman, Rizzolo, Slivken
Brownian excursion



$Av(231)$

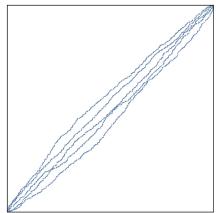
Traceless Dyson
Brownian bridge



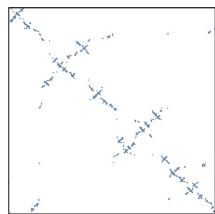
$Av(321)$

Bossino, Bouvel, Féray, Gerin, Maazoun, Pierrot
Continuum Random Tree

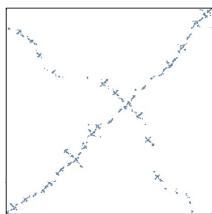
Borga & Maazoun
flows of SDEs + LQG



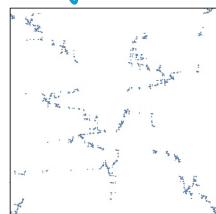
$Av(654321)$



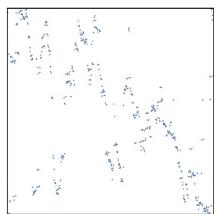
$Av(2413, 3142)$



$SE(Av(321))$



Baxter



Semi-Baxter

On the space of permutoons we have a natural notion of CONVERGENCE.

Let \mathcal{M} be the space of permutoons. We say that a sequence of permutoons $(\mu_n)_n$ converges (in the permutoon sense) if

$$\int_{[0,1]^2} f \, d\mu_n \xrightarrow{n \rightarrow \infty} \int_{[0,1]^2} f \, d\mu \quad \forall \text{ (bounded) \& cont. function } f: [0,1]^2 \rightarrow \mathbb{R}$$

With this topology \mathcal{M} is a compact metric space.

$$\hookrightarrow d_\square(\mu, \nu) = \sup_{R \in \mathcal{R}} |\mu(R) - \nu(R)|$$

(For a proof see Lemmas 2.5 & 5.3 of Hoppen, Kohayakawa, Moreira, Roth, Sampaio
("Limits of permutation sequences".))

History: Permutons were introduced by Hoppen, Kohayakawa, Moreira, Roth, Sampaio

History: Permutons were introduced by Hoppen, Kohayakawa, Moreira, Roth, Sampaio in "Limits of permutation sequences" employing a different but equivalent definition.

The measure theoretic view presented above was originally used by Presutti and Stromquist in "Packing rates of measures and a conjecture for the packing density of 2413" and was later exploited by Glebov, Grzesik, Klímošová, Král' in "Finitely forcible graphons and permutons" in which the term "permuto" was first used.

Another paper that contributed to develop the theory of permutons is "Permutations with fixed pattern densities" by Kenyon, Král', Radin, Winkler.

Random permutons were deeply studied by Bossino, Bouvel, Féray, Gerin, Maazoun, Piernot in "Universal limits of substitution-closed permutation classes".

2.2 Random permutons

we denote random quantities
using **BOLD** characters

Def: A random permuto \mathbf{m} is a random variable from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to the space of permutons M .

Example: A typical example of random permuto is \mathbf{m}_σ where σ is a random permutation.

There is a nice characterization of permuto convergence in terms of convergence of proportion of occurrences of patterns:

THEOREM: $\forall n \in \mathbb{Z}_{>0}$, let σ_n be a random permutation of size n .

Moreover, for every fixed K , let $I_{n,K}$ be a uniform random subset of $[n]$ of cardinality K , independent of σ_n . TFAE:

(a) $m_{\sigma_n} \xrightarrow[n \rightarrow \infty]{d} m$ in the permutoon sense;

(b) $(\widetilde{\text{occ}}(\pi, \sigma_n))_{\pi \in S} \xrightarrow{d} (\Delta_\pi)_{\pi \in S}$ w.r.t. the product topology;

(c) For every $\pi \in S$, $\exists \Delta_\pi > 0$ s.t. $\mathbb{E}[\widetilde{\text{occ}}(\pi, \sigma_n)] \xrightarrow{n \rightarrow \infty} \Delta_\pi$.

(d) For every K , $\text{pat}_{I_{n,K}}(\sigma_n) \xrightarrow{d} f_K$.

Moreover, if these assertions are verified, then

$$(\Delta_\pi)_{\pi \in S} \stackrel{d}{=} (\widetilde{\text{occ}}(\pi, m))_{\pi \in S} \quad \begin{matrix} \text{see below for} \\ \text{definition.} \end{matrix}$$

and

$$P(f_K = \pi) = \Delta_\pi = \mathbb{E}[\Delta_\pi] = \mathbb{E}[\widetilde{\text{occ}}(\pi, m)].$$

Ideas of proof:

- (c) \Leftrightarrow (d): Simple consequence of

$$P(\text{pat}_{I_{n,K}}(\sigma_n) = \pi) = \mathbb{E}[\widetilde{\text{occ}}(\pi, \sigma_n)]$$

- (a) \Rightarrow (b): Define $\widetilde{\text{occ}}(\pi, m) := P(\underbrace{U_1, \dots, U_K}_{K \text{ points sampled according to } m} \text{ forms the pattern } \pi | m)$

Then the result follows from the fact that for every deterministic permutoon m the map $m \mapsto \widetilde{\text{occ}}(\pi, m)$ is continuous & from the following result:

Lemma 1: If $\pi \in S_K$ and $\sigma \in S_n$ then

$$|\widetilde{\text{occ}}(\pi, \sigma) - \widetilde{\text{occ}}(\pi, m_\sigma)| \leq \frac{1}{n} \binom{K}{2}.$$

- (d) \Rightarrow (a) Tells us that the permutoon m is well-approximated

- (d) \Rightarrow (e) [Tells us that the permuton μ_{σ_n} is well-approximated by $\text{pat}_{I_{n,x}}(\sigma_n)$]. This implication is a consequence of the following:

Lemma 2: (Approximation of a random permuton by a random permutation)

There exists K_0 such that if $K > K_0$

$$P(d_{\square}(\mu_{\text{Perm}_K(Y)}, \nu) \geq 16K^{-1/4}) \leq \frac{1}{2} e^{-\sqrt{K}} \quad \forall \text{ random permuton } \nu$$

\hookrightarrow "Permutation determined by K random points in $[0,1]^2$ sampled according to Y ."

The lemma above together with

$$P(\text{Perm}_K(\mu_{\sigma_n}) = \pi) \sim P(\text{pat}_{I_{n,x}}(\sigma_n) = \pi) \quad (\text{Consequence of Lemma 1})$$

are enough to conclude.

- (c) \Rightarrow (b) is SURPRISING: Convergence in expectation is enough!

The reason why this is true comes from the following deterministic result:

CLAIM: Fix patterns π_1, \dots, π_K . There exist constants c_p such that for all permutations $\sigma \in S$,

$$\prod_{i=1}^K \text{occ}(\pi_i, \sigma) = \sum_{p \in S} c_p \text{occ}(p, \sigma).$$

\hookrightarrow (For a proof, see for instance Theorem 1.4 of Penaguiao, "Pattern Hopf algebras")

Therefore joint moments of $\tilde{\text{occ}}(\pi_i, \sigma_n)$ are linear combinations of expectations. If the expectations converge, then the joint moments converge and we have convergence in distribution. \square