Home work V

giovedì 29 ottobre 2020 18:07

Exercise 1

(a) $f \in M(\mathbb{R}^n, \mathbb{R}) \Rightarrow |f| \in M(\mathbb{R}^n, \mathbb{R})$

Solution: We first prove that $\forall \varphi \in T(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ then

$$(|f|)|_\varphi = |f|_\varphi|$$

By definition

$$(|f|)|_\varphi = \max\{-\varphi, \min\{|f|, \varphi\}\} = \min\{|f|, \varphi\}$$

$$|f|_\varphi| = |\max\{-\varphi, \min\{f, \varphi\}\}|$$

It is easy to see that $\min\{|f|, \varphi\} \overset{\circledast}{=} |\max\{-\varphi, \min\{f, \varphi\}\}|$ by distinguishing 3 cases: $f \leq -\varphi$, $-\varphi < f < \varphi$, & $f \geq \varphi$.

Now since $f|_\varphi \in L^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$, by 10.2.14(3) $|f|_\varphi| \in L^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$

$\overset{\circledast}{\Rightarrow} (|f|)|_\varphi \in L^1(\mathbb{R}^n, \mathbb{R}_{\geq 0}) \Rightarrow |f| \in L^1(\mathbb{R}_n, \mathbb{R}_{\geq 0})$ by 10.4.2(2).

(b) Are $f(x) = \begin{cases} 1/x & x \in \mathbb{R} \setminus \{0\} \\ 0 & x = 0 \end{cases}$ & $g(x) = \begin{cases} \sin(1/x) & x \in \mathbb{R} \setminus \{0\} \\ 0 & x = 0 \end{cases}$

By prop. 10.4.3 (4) it is enough to show that $\exists (f_k)_k$ & $(g_k)_k$ in $\mathcal{M}$ s.t. $f_k \xrightarrow{a.e.} f$ & $g_k \longrightarrow g$

Take

$$f_k(x) := f(x)\,\chi_{\mathbb{R} \setminus (-\frac{1}{n}, \frac{1}{n})} + n^2 x\, \chi_{[-\frac{1}{n}, \frac{1}{n}]}(x) \in \mathcal{C} \underset{\uparrow}{\subseteq} M$$

10.4.3(2)

$$g_k(x) := g(x)\,\chi_{\mathbb{R}\setminus(-\frac{1}{n},\frac{1}{n})} + n\sin(n)\,x\,\chi_{[-\frac{1}{n},\frac{1}{n}]} \in \mathcal{C} \subseteq M \qquad \square$$

Exercise 2:

$$f \in L^1(\mathbb{R}^n,\mathbb{R}) \Rightarrow f_+, f_- \in L^1(\mathbb{R}^n,\mathbb{R})$$

Solution:

Since L^1 is a vector space $-f \in L^1$.

By HW 3, ex 1, part b, $\max\{f,0\}$ & $\max\{-f,0\} \in L^1$.

Exercise 3

(a) $$\int_0^n \underbrace{\sin(nx)}_{g'}\, \underbrace{e^{-x}}_{f}\, dx = \left[\left(-\frac{\cos(nx)}{n}\right)e^{-x}\right]_{x=0}^{n} - \int_0^n \left(-\frac{\cos(nx)}{n}\right)\left(-e^{-x}\right) dx$$

$$= -\frac{\cos(n^2)}{n}e^{-n} - \frac{1}{n} - \frac{1}{n}\underbrace{\int_0^n \cos(nx)\,e^{-x}\,dx}_{\left[\frac{\sin(nx)}{n}e^{-x}\right]_{x=0}^{n} + \int_0^n \frac{\sin(nx)}{n}e^{-x}}$$

$$= -\frac{\cos(n^2)}{n}e^{-n} - \frac{1}{n} - \frac{1}{n}\left(\frac{\sin(n^2)}{n}e^{-n}\right) - \frac{1}{n}\int_0^n \frac{\sin(nx)}{n}e^{-x}\,dx$$

If $I_n = \int_0^n \sin(nx)\,e^{-x}\,dx$ then

$$I_n = -\frac{e^{-n}}{n}\left(\cos(n^2) + \frac{\sin(n^2)}{n}\right) - \frac{1}{n} - \frac{1}{n^2} I_n$$

$$\Rightarrow \frac{I_n}{n} = \left(-\underset{\downarrow\,0}{e^{-n}}\left(\underset{\leq 1}{\cos(n^2)} + \underset{\downarrow\,0}{\frac{\sin(n^2)}{n}}\right) - \underset{\downarrow\,0}{\frac{1}{n}}\right)\underset{\downarrow\,0}{\frac{n}{n^2+1}} \longrightarrow 0$$

(b) $$\lim_{n\to\infty} \int_0^n \left(1-\frac{x}{n}\right)^n dx = ?$$

$$\int_0^n \left(1-\frac{x}{n}\right)^n dx = (-n)\int_0^n \left(1-\frac{x}{n}\right)^n \left(-\frac{1}{n}\right) dx =$$

$$= (-n)\left[\left(1-\frac{x}{n}\right)^{n+1}\frac{1}{n+1}\right]_{x=0}^{n} = -\frac{n}{n+1}(0-1) = \frac{n}{n+1} \longrightarrow 1$$

(c) $$f_n(x) := \underbrace{\left(1+\frac{x}{n}\right)^n x\, e^{-2x}\, \chi_{[0,n]}(x)}_{\in L^1 \text{ (cont. on compact.)}} \xrightarrow[n\to\infty]{a.e.} e^{-x} x\, \chi_{[0,+\infty)}(x)$$

$$|f_n(x)| \leq \underbrace{e^{-x} x\, \chi_{[0,+\infty)}(x)}_{\in L^1 \text{ (meas. + finite int.)}}$$

By dom. conv. thm.

$$\int f_n(x)\,dx \xrightarrow[n\to\infty]{} \int e^{-x} x\, dx = 1.$$ □

Exercise 4:

Solution:

$$f = \underbrace{f\chi_{[\frac{1}{k},1]}}_{g \in L^{inc}} - \underbrace{\left(-f\chi_{(0,\frac{1}{k})}\right)}_{h \in L^{inc}}$$

(cont. on comp) 10.2.16 (see 10.2.9.)

The rest is like last week.

$y=2$

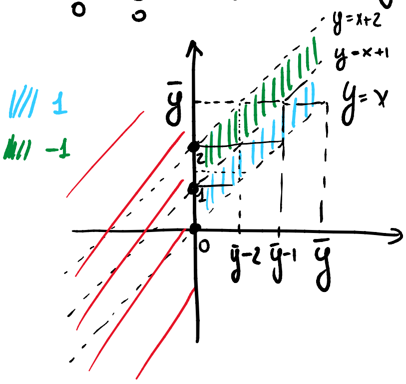
$x \geq 0,\ x \leq y < x+1$

$x \geq 0,\ x+1 \leq y < x+2$

Exercise 5:

$$\int_0^{+\infty}\int_0^{+\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^y 1\,dx\,dy + \int_1^2\left(\int_0^{y-1} -1\,dx + \int_{y-1}^{y} +1\,dx\right)dy + \int_2^{+\infty}\left(\int^{y-1} -1\,dx + \int^{y} 1\,dy\right)dx =$$

$$\int_0 \int_0 f(x,y)\, dx\, dy = \int_0 \int_0 1\, dx\, dy + \int_1 \left(\int_0 -1\, dx + \int_{y-1} +1\, dx \right) dy + \int_2 \left(\int_{y-2}^{0} -1\, dx + \int_{y-1}^{0} 1\, dy \right) dx =$$



$$= \int_0^1 y\, dy + \int_1^2 -y + 1 + (y - y + 1)\, dy + \int_2^{+\infty} (-1)\cdot(y-1-y+2) + 1\cdot(y-y+1)\, dx$$

$$= \int_0^1 y\, dy + \int_1^2 -y+2\, dy + \int_2^{+\infty} 0\, dx$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

$$\int_0^{+\infty} \left(\int_0^{+\infty} f(x,y)\, dy \right) dx = \int_0^{+\infty} \left(\int_x^{x+1} 1\, dy + \int_{x+1}^{x+2} (-1)\, dy \right) dx$$

$$= \int_0^{+\infty} (1 - 1)\, dx = 0$$

Noting that

$$\int |f(x,y)|\, dy = 2\chi_{[0,+\infty)}(x) \notin L^1$$

$$\int |f(x,y)|\, dx = y\,\chi_{[0,2]} + 2\cdot\chi_{[2,+\infty)}(y) \in L^1$$

we can conclude that this is <u>not</u> a counterexample to Tonelli