Homework VI

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Exercise 1

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ cont. & $U = f^{-1}(]-1,1[)$

Solution:

(a) U is open because f is continuous. By 10.4.9.(1) then U is measurable.

(b) The claim is false. For instance if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ s.t. $x \longmapsto \frac{1}{2}$ $\forall x \in \mathbb{R}^n$ then $U = \mathbb{R}^n$ & $\lambda_n(U) = +\infty$.

Exercise 5

Let $\Omega =]0,1[$, $f \in L^1(\Omega)$ and define

$$F(x) = \int_0^x f(s)\,ds \qquad \forall x \in \Omega$$

Show that $F(x) \in L^1(\Omega)$

Proof: By definition 10.2.13(1), $F \in L^1$ if $\exists$ $G, H \in L^{inc}$ s.t. $F = G - H$. The goal is to determine these G, H.

Since $f \in L^1$ then $\exists$ $g, h \in L^{inc}$ s.t. $f = g - h$. Moreover $\exists$ $(g_k)_k, (h_k)_k \in T^{inc}$ s.t. $g_k \xrightarrow{a.e.} g$, $h_k \xrightarrow{a.e.} h$ & $(\int g_k)_k, (\int h_k)_k$ are bounded (*)

We now fix $x \in \Omega$ and define

$$g_k^x(s) := g_k(s)\, \chi_{[0,x]}(s)$$
$$h_k^x(s) := h_k(s)\, \chi_{[0,x]}(s)$$

then $g_k^x \xrightarrow{a.e.} g\,\chi_{[0,x]}$, $h_k^x \xrightarrow{a.e.} h\,\chi_{[0,x]}$ & $(\int g_k^x)_k, (\int h_k^x)_k$ are bounded.

Moreover, by definition 10.2.11.

$$\int_0^x g(s)\,ds = \int_0^1 g(s)\chi_{[0,x]}(s)\,ds = \lim_{k\to\infty}\int_0^1 g_k^x(s)\,ds \qquad (1)$$

$$\int_0^x h(s)\,ds = \int_0^1 h(s)\chi_{[0,x]}(s)\,ds = \lim_{k\to\infty}\int_0^1 h_k^x(s)\,ds \qquad (2)$$

Now we define

$$G_k(x) := \int_0^1 g_k^x(s)\,ds \qquad (3)$$

$$H_k(x) := \int_0^1 h_k^x(s)\,ds \qquad (4)$$

We want to show that:

$$\left.\begin{array}{l}
(1)\quad G_k(x),\ H_k(x) \in L^{inc} \\
(2)\quad \left(\int_0^1 G_k(x)\,dx\right)_k \ \&\ \left(\int_0^1 H_k(x)\,dx\right) \text{ are bounded} \\
\text{Then by } (1)+(2):\ G_k(x) \xrightarrow{a.e.} G(x),\quad H_k(x) \xrightarrow{a.e.} H(x)
\end{array}\right\} \Rightarrow G, H \in L^{inc} \qquad (5)$$

(4) $F(x) = G(x) - H(x)$

We show the first 2 points only for G_k, G:

(1) We start by showing the hint:

Lemma: I interval in Ω, then $G(x) := \int_0^x \chi_I(s)\,ds \in L^{inc}$

Proof: $\chi_I \in S([0,1],\mathbb{R})$. Therefore, by 7.2.8, $G \in \mathcal{C}([0,1],\mathbb{R})$.

↳ step-continuous functions

So by 10.2.16(2), $G \in L^{inc}$. $\square$

Using the lemma above and the fact that

$$G_k(x) = \int_0^x g_k(s)\,ds$$

↳ step function

we can conclude that $G_k(x) \in L^{inc}$

$\hookrightarrow$ step function

we can conclude that $G_k(x) \in L^{inc}$.

$$(2) \quad \int_0^1 G_k(x)\,dx = \int_0^1 \underbrace{\int_0^x g_k(s)\,ds}_{\leq \int_0^1 g_k(s)ds}\,dx \leq \int_0^1\left(\int_0^1 g_k(s)ds\right)ds \leq \int_0^1 g_k(s)\,ds$$

$\Rightarrow \left(\int_0^1 G_k(x)\,dx\right)$ is bounded because $\left(\int_0^1 g_k\right)_k$ is bounded by (*).

It remains to prove that $F(x) = G(x) - H(x)$

Note that

$$F(x) = \int_0^x f(s)\,ds = \int_0^x (g(s) - h(s))ds$$

$$= \int_0^x g(s)\,ds - \int_0^x h(s)\,ds$$

$$\overset{(1)+(2)}{=} \lim_{k\to\infty} \int_0^x g_k(s)\,ds - \int_0^x h_k(s)\,ds$$

$$\overset{(3)+(4)}{=} \lim_{k\to\infty} G_k(x) - H_k(x) \overset{(5)}{=} G(x) - H(x) \qquad \square$$

Exercise 2

$\Omega = [a,b]$, $a<b \in \mathbb{R}$, $f \in \mathcal{C}^1$, $f'>0$. Then $f(\Omega)$ is meas. with bounded Leb. meas.

Solution:

Ω is compact since closed & bounded.

If f is continuous, then $f(\Omega)$ is compact too (see 5.1.23 (proof))

But if $f(\Omega)$ is compact then $f(\Omega) \in \mathcal{M}$ & $\lambda_n(f(\Omega)) < \infty$

(see 10.4.9. (3))

Exercise 3

Take $X = \mathbb{R}$ and $\mathcal{A} = \{\emptyset, \mathbb{Q}, \mathbb{R}\setminus\mathbb{Q}, \mathbb{R}\}$ and

$$\mu(\emptyset) = 0$$
$$\mu(\mathbb{Q}) = \mu(\mathbb{Q}\setminus\mathbb{R}) = \frac{1}{2}$$
$$\mu(\mathbb{R}) = 1$$

Then $(X, \mathcal{A}, \mu)$ is a probability space. Indeed (see def. 10.4.8)

$\mathcal{A}$ is a σ-algebra:

- $\emptyset, X \in \mathcal{A}$
- if $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
- if $(A_k)_k \in \mathcal{A}$ then $\bigcup_k A_k \in \mathcal{A}$ (distinguish various cases....)

μ is a proba:

- $\mu(\emptyset) = 0$
- $(A_k)_k \subseteq \mathcal{A}$ disjoint $\Rightarrow \mu\left(\bigcup_k A_k\right) = \sum_k \mu(A_k)$ (again distinguish cases)

Exercise 4

$a, b \in \mathbb{R}$, $f, g : [a,b] \to \mathbb{R}$ continuous, $0 \leq f \leq g$

$$A := \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, f(z) \leq \sqrt{x^2+y^2} \leq g(z),\ a \leq z \leq b \right\}$$

is meas with finite measure such that:

$$\lambda_3(A) := \pi \int_a^b \left(g(z)^2 - f(z)^2 \right) dz$$

Solution:

$$A = \underbrace{\left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, f(z) - \sqrt{x^2+y^2} \leq 0 \right\}}_{= F^{-1}((-\infty, 0])} \cap \underbrace{\left\{ (x,y,z) \in \mathbb{R}^2 \,\middle|\, g(z) - \sqrt{x^2+y^2} \geq 0 \right\}}_{G^{-1}([0, +\infty))} \cap \underbrace{\left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, a \leq z \leq b \right\}}_{\text{closed}}$$

where $F(x,y,z) = f(z) - \sqrt{x^2+y^2}$

where

$= F^{-1}((-\infty,0])$ where $F(x,y,z)=f(z)-\sqrt{x^2+y^2}$ is continuos

$G^{-1}([0,+\infty))$ where $G(x,y,z)=g(z)-\sqrt{x^2+y^2}$ is cont.

Since the preimage of a closed set via a cont. function is closed we can conclude that A is the intersection of 3 closed sets and so it is closed.

Moreover if $M=\max_{[a,b]}\{g\}$, $m=\min_{[a,b]}\{f\}$ then

$$A \subseteq \left\{ (x,y,z)\in\mathbb{R}^3 \;\middle|\; m\le\sqrt{x^2+y^2}\le M,\ a\le z\le b \right\} \to \underline{\underline{\text{bounded!}}}$$

$\Rightarrow$ A is bouded and closed $\Rightarrow$ A is compact $\Rightarrow$ A is meas. & $\lambda_3(A)<\infty$.

Note that $A=B\setminus D$ with

$$B=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \sqrt{x^2+y^2}\le g(z),\ a\le z\le b\right\}$$

$$D=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \sqrt{x^2+y^2}< f(z),\ a\le z\le b\right\}$$

$$\Rightarrow \lambda_3(A)=\lambda_3(B)-\lambda_3(D)=\int_{\mathbb{R}^3}\chi_B-\int_{\mathbb{R}^3}\chi_D$$

Note that

$$\int_{\mathbb{R}^3}\chi_B \overset{\text{Fubini}}{=} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{[0,g(z)]}\left(\sqrt{x^2+y^2}\right)\chi_{[a,b]}(z)\,dx\,dy\,dz =$$

determinant of Jacobian

$$\overset{!}{=} \int_a^b\left(\int_0^{2\pi}\int_0^{+\infty}\chi_{[0,g(z)]}(r)\cdot r\,dr\,d\theta\right)dz$$

polar coordinates
$r=\sqrt{x^2+y^2}$
$\theta=\arctan(y/x)$

$$\overset{!}{=} \int_a^b\int_0^{2\pi}\frac{g(z)^2}{2}\,d\theta\,dz=\int_a^b\pi\,g(z)^2\,dz$$

Similarly we can compute $\int_{\mathbb{R}^3} \chi_D$. $\square$