Homework VII

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Exercise 1

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$.

$$f \text{ is measurable } \overset{?}{\Longleftrightarrow} |f| \text{ is measurable.}$$

Solution:

($\Longrightarrow$) True & it was proved in HW5.

($\Longleftarrow$) FALSE. Let's find a counterexample:

Let $M \subseteq \mathbb{R}$ be the not measurable set constructed in example 10.4.10 of the lecture notes.

Now set

$$f(x) = \chi_M(x) - \chi_{\mathbb{R}\setminus M}$$

$f^{-1}(\{1\}) = M$ not meas.

It is easy to realize that f is NOT measurable but

$$|f| \equiv 1 \quad \text{is measurable} \qquad \square$$

Exercise 2

This is a lemma for proving Thm 10.5.1 (Transformationsatz)

We want to complete the proof of Satz 10.5.4 (Step 4).

$I \subseteq \mathbb{R}^n$ bnd interval, $\sigma : x \longmapsto S_{ij}\, x \qquad 1 \leq i \neq j \leq n$.

We want to show:

$$\lambda_n(\sigma(I)) = |\det S_{ij}|\, \lambda_n(I).$$

Solution:

$\vec{x} = \begin{pmatrix} x_1 \end{pmatrix}$ $\begin{pmatrix} 1 \end{pmatrix}$

Solution:

$$S_{ij} = \begin{array}{cc} \begin{pmatrix} 1 & & t \\ & \ddots & \\ & & 1 \end{pmatrix} & i \\ j & \end{array} \overset{x}{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = \vec{x} + t x_j \cdot \underbrace{\vec{e}_i}_{= \begin{array}{cc}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{matrix} \\ \\ \\ i \\ \\ \\ \\ \end{matrix}\end{array}} = \sigma(\vec{x})$$

$\det(S_{ij}) = 1$ because S_{ij} is upper-triangular (lower-triang.).

Therefore it is enough to show that $\lambda_n(\sigma(I)) = \lambda_n(I)$.

Assume

$$I = [a_1, b_1] \times \dots \times [a_n, b_n] \quad \text{for some } a_i \le b_i \in \mathbb{R}$$

Therefore $\lambda_n(I) = \prod_{i=1}^{n} (b_i - a_i)$.

Now we want to compute

$$\lambda_n(\sigma(I)) = \int_{\mathbb{R}^n} \chi_{\sigma(I)}(\vec{x})\, d\vec{x} = \int_{\mathbb{R}^n} \chi_{\underset{[a_1,b_1] \times \dots \times [a_n,b_n]}{I}}(\vec{x} - t x_j \vec{e}_i)\, d\vec{x} = \circledast$$

$$\chi_{\sigma(I)}(\vec{x}) = 1 \iff \vec{x} \in \sigma(I) \iff \vec{x} - t x_j \vec{e}_i \in I \iff \chi_I(\vec{x} - t x_j \vec{e}_i) = 1$$

$$\sigma(\vec{x}) = \vec{x} + t x_j \vec{e}_i$$

$$\sigma^{-1}(\vec{x}) = \vec{x} - t x_j \vec{e}_i$$

$$\circledast = \int_{\mathbb{R}^n} \chi_{[a_1,b_1]}(x_1) \cdots \underline{\chi_{[a_i,b_i]}(x_i - t x_j)} \cdots \chi_{[a_n,b_n]}(x_n)\, dx_1 \dots dx_n$$

Note that $\int_{\mathbb{R}} \chi_{[a_i,b_i]}(x_i - t \overset{\text{const.}}{x_j})\, dx_i = \big(b_i - t x_j - (a_i - t x_j)\big) = b_i - a_i$

and so

$$\circledast = \prod_{i=1}^{n} (b_i - a_i) = \lambda_n(I)$$ $\square$

Exercise 3

Compute the integral

$$\int_D \frac{2x_1^2 + x_2}{x_1 x_2} \, dx_1 dx_2$$

where $D = \{ x \in \mathbb{R}^2 \mid 1 < x_1 x_2 < 4, \; 0 < x_2 - x_1^2 < 1 \}$

Solution:

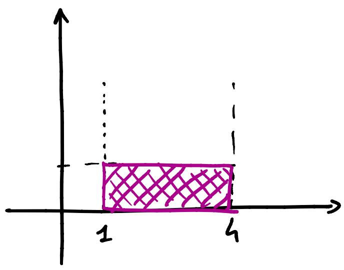
$$D = \{ x \in \mathbb{R}^2 \mid 1 < x_1 x_2 < 4, \; 0 < x_2 - x_1^2 < 1 \}$$

IDEA: Change variables!

Set $u(x_1, x_2) = x_1 \cdot x_2$ & $v(x_1, x_2) = x_2 - x_1^2$

Then

$$E = \{ (u,v) \in \mathbb{R}^2 \mid 1 < u < 4, \; 0 < v < 1 \} = \,]1,4[\times]0,1[$$



Moreover $\psi : (x_1, x_2) \longrightarrow (\overset{u}{\overset{\shortparallel}{x_1 x_2}}, \overset{v}{\overset{\shortparallel}{x_2 - x_1^2}})$ is a $\mathcal{C}^1$-diffeo (exercise)

We are going to use the map ψ^{-1}. Note that

$$\left| \det D\psi^{-1}(x_1, x_2) \right| = \frac{1}{\left| \det D\psi(x_1, x_2) \right|} = \frac{1}{x_2 + 2x_1^2}$$

$$= \begin{pmatrix} x_2 & x_1 \\ -2x_1 & 1 \end{pmatrix}$$

Therefore:

$$\int_D \frac{2x_1^2 + x_2}{x_1 x_2}\, dx_1 dx_2 \overset{\text{Transformationsatz}}{=} \int_E \frac{\cancel{2\, x_1^2(u,v) + x_2(u,v)}}{\underbrace{x_1(u,v)\, x_2(u,v)}_{u}} \, \frac{1}{\cancel{x_2(u,v) + 2x_1^2(u,v)}} \, du dv$$

$$= \int_E \frac{1}{u}\, du dv = \int_1^4 \int_0^1 \frac{1}{u}\, dv\, du = \int_1^4 \frac{1}{u} du = \ln(4) \quad \square$$

Exercise 4

(a) $\kappa: \mathbb{R}_{>0} \times [0, 2\pi[\times [0,\pi] \longrightarrow \mathbb{R}^3 \setminus \{0\}$ is a C^1-diffeo

$$(r, \phi, \theta) \longrightarrow (r \sin\theta \cos\phi,\ r \sin\theta \sin\phi,\ r\cos\theta)^T$$

Let $B_3(0,1) = \{ x \in \mathbb{R}^3 \mid \|x\|_2 < 1 \}$. For which $s \in \mathbb{R}$ is

$$f: B_3(0,1) \longrightarrow \mathbb{R}, \quad f(x) := \|x\|_2^s \quad \text{integrable?}$$

Solution:

$$\kappa(r, \phi, \theta) = (r \sin\theta \cos\phi,\ r \sin\theta \sin\phi,\ r\cos\theta)$$

$$|\det D_\kappa(r,\phi,\theta)| = \left| \det \begin{pmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi & r\cos\phi\cos\theta \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\sin\phi\cos\theta \\ \cos\theta & 0 & -r\sin\theta \end{pmatrix} \right|$$

$$= \ldots\ldots = |r^2 \sin\theta|$$

r^s T.S.+F.T.

$1 \quad 2\pi \quad \pi$

$$\int_{B_3} \overbrace{\|\vec{x}\|_2^s}^{r^s} \, d\vec{x} \overset{TS + T.1.}{=} \int_0^1 \int_0^{2\pi} \int_0^{\pi} r^s \cdot r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= 2\pi \int_0^1 r^{2+s} \, dr \underbrace{\int_0^{\pi} \sin\theta \, d\theta}_{=2} = 4\pi \underbrace{\int_0^1 r^{2+s} \, dr}_{< \infty}$$

$$\int_0^1 \frac{1}{r^{-2-s}} \, dr$$

$$\Updownarrow$$

$$2 + s > -1$$

$$\Updownarrow$$

$$\boxed{s > -3}$$

We can conclude that

$$\int_{B_3} \|x\|_2^s \, dx \text{ is int. if and only if } s > -3. \qquad \square$$

(b) $K: [0,r] \times [0, 2\pi[\times [0, 2\pi[\longrightarrow T$ $\quad$ $\mathcal{C}^1$-diffeo (ex.)

$$(s, \phi, \theta) \longmapsto \left(\cos\phi \, (R + s\cos\theta), \sin\phi \, (R + \cos\theta), s . \sin\theta\right)$$

By the TS:

$$\int_{\mathbb{R}^3} \chi_T(x) \, dx = \int_T 1 \, dx = \int_0^r \int_0^{2\pi} \int_0^{2\pi} \underbrace{\left|\det D_K(s, \phi, \theta)\right|}_{= Rs + s^2\cos\theta} \, d\theta \, d\phi \, ds$$

$$= \int_0^r \int_0^{2\pi} \int_0^{2\pi} Rs + s^2 \cos\theta \, d\theta \, d\phi \, ds = (\text{standard comp.}) = 4\pi^2 R r^2 .$$

Exercise 5

(a) $f(x) = \frac{1}{x}$ works!

(b) $g : \,]0,1[\, \longrightarrow \mathbb{R}$ $\qquad g(x) = \dfrac{1}{x(1-\log(x))^2}$

We want to show that

$$g^s \in L^1 \iff s \leq 1.$$

<u>Solution</u>: g is measurable because it is continuous in $]0,1[$.

$\boxed{s=1}$

$$\int_0^1 \frac{1}{x(1-\log(x))^2}\, dx = \ldots\ldots = \left[\frac{1}{1-\log(x)}\right]_{x=0}^{1} = 1 < +\infty.$$

$$\Rightarrow g \in L^1$$

$\boxed{s>1}$ Let $s = 1+2\delta$ for some $\delta > 0$.

$$(g(x))^s = \frac{1}{\underbrace{x^s}_{= x^{1+2\delta}} (1-\log(x))^s} = \frac{1}{x^{1+\delta}} \underbrace{\frac{1}{x^{\delta}(1-\log(x))^{2+4\delta}}}_{= h(x)}$$

$$h(x) \geq C > 0 \Longleftarrow \begin{cases} \cdot\ h(x) \xrightarrow[x \to 0^+]{} +\infty \\ \cdot\ h(1) = 1 \\ \cdot\ h \text{ is continuous } \& \ h > 0 \end{cases}$$

$$\Rightarrow g(x)^s \geq \frac{1}{x^{1+\delta}} \cdot C$$

$$\Rightarrow \int_0^1 (g(x))^s\, dx \geq C \underbrace{\int_0^1 \frac{1}{x^{1+\delta}}\, dx}_{+\infty} \geq +\infty \Rightarrow g(x)^s \notin L^1 \text{ if } s > 1$$

$\boxed{s<1}$ Now $s = 1-2\delta$ for some $\delta > 0$.

$$(g(x))^s = \frac{1}{x^{1-\delta}} \quad \frac{1}{x^{-\delta}\,(1-\log(x))^{2-4\delta}}$$

$$= \frac{x^{\delta}}{(1-\log(x))^{2-4\delta}} =: h(x)$$

$$\left.\begin{array}{l} \bullet\ h(x) \xrightarrow[x \to 0^+]{} 0 \\ \bullet\ h(1) = 1 \\ \bullet\ h \text{ is cont.} \end{array}\right\} \Rightarrow \begin{array}{l} \exists\, C>0 \text{ s.t.} \\ h(x) \leq C \end{array}$$

$$\Rightarrow (g(x))^s \leq \frac{C}{x^{1-\delta}}$$

$$\Rightarrow \int_0^1 (g(x))^s\,dx \leq C \underbrace{\int \frac{1}{x^{1-\delta}}\,dx}_{<\infty} < \infty$$

$$\Rightarrow (g(x))^s \in L^1 \quad \text{for} \quad s < 1 . \qquad \square$$