Homework XI

venerdì 4 dicembre 2020 14:44

Exercise 1

(a) Define $\gamma : [0,2] \longrightarrow \mathbb{R}^2$, $x \mapsto (x, x^2)$ then

$$\int_\gamma f \, dx = \int_0^2 \langle f \cdot \gamma, \gamma' \rangle = \int_0^2 \left\langle \begin{pmatrix} 2x\cos(2x^2) \\ \cos(2x^2) \end{pmatrix}, \begin{pmatrix} 1 \\ 2x \end{pmatrix} \right\rangle = \int_0^2 4x \cos(2x^2)\, dx = \int_0^8 \cos(y)\, dy = \sin(8)$$

(b) Define $\gamma : [0,1] \longrightarrow \mathbb{R}$, $x \mapsto \begin{pmatrix} 2x \\ 4x \end{pmatrix}$ then

$$\int_\gamma f \, dx = \int_0^1 \left\langle \begin{pmatrix} 4x^2 \\ 16x^2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\rangle dx = \int_0^1 72\, x^2 \, dx = 24$$

(c) Define $\gamma_1 : [0,1] \to \mathbb{R}^2$, $x \mapsto \begin{pmatrix} 2x \\ 0 \end{pmatrix}$, $\gamma_2 : [0,1] \to \mathbb{R}^2$, $x \mapsto \begin{pmatrix} 2 \\ x \end{pmatrix}$, $\gamma_3 : [0,1] \to \mathbb{R}^2$, $x \mapsto \begin{pmatrix} 2-2x \\ 1 \end{pmatrix}$, $\gamma_4 : [0,1] \to \mathbb{R}^2$, $x \mapsto \begin{pmatrix} 0 \\ 1-x \end{pmatrix}$

Then
$$\int_{\gamma = \gamma_1+\gamma_2+\gamma_3+\gamma_4} f \, dx = e^2 \int_0^1 x \, dx + \int_0^1 (4x-4)\, dx + \int_0^1 (x-1)\, dx = \underline{\frac{e^2-5}{2}}$$

(d) Define $\gamma : [0,2\pi] \longrightarrow \mathbb{R}^3$, $x \mapsto \begin{pmatrix} \sin(x) \\ \cos(x) \\ x \end{pmatrix}$ then

$$\int_\gamma f \, dx = \int_0^{2\pi} \left\langle \begin{pmatrix} \sin^2(x) + 5\cos(x) + 3x\cos(x) \\ 5\sin(x) + 3x\sin(x) - 2 \\ 3\sin(x)\cos(x) - 4x \end{pmatrix}, \begin{pmatrix} \cos(x) \\ -\sin(x) \\ 1 \end{pmatrix} \right\rangle dx$$

$$\stackrel{!}{=} -8\pi^2$$

Exercise 2

Exercise 2

$\Upsilon = f^{-1}(\{0\})$ with $f(x) = \left(\left(\sqrt{x_1^2+x_2^2} - R\right)^2 + x_3^2 - r^2 \right)$

(a) Clearly $f \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R})$.

$$D_f(x) = \left(2\left(\sqrt{x_1^2+x_2^2} - R\right) \frac{x_1}{\sqrt{x_1^2+x_2^2}} , \; 2\left(\sqrt{x_1^2+x_2^2} - R\right) \frac{x_2}{\sqrt{x_1^2+x_2^2}} , \; 2x_3 \right)$$

If $D_f(x) = 0$ then $x_3 = 0$

If $f(x) = 0$ & $x_3 = 0$ then $\left|\sqrt{x_1^2+x_2^2} - R\right| = r$

Therefore if $f(x) = 0$ then $D_f(x) \neq (0,0,0)$.

Hence $rk(D_f(x)) = 1$ $\forall x \in \Upsilon$ and so Υ is a 2-dim. $\mathcal{C}^\infty$ manifold.

(b) We start by showing that $\Psi(\alpha, \beta)$ is an homeom.:

- Ψ is obviously continuous.
- Injectivity: see inverse
- Surjectivity: Suppose $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in f^{-1}(\{0\})$ i.e. $\left(\sqrt{x_1^2+x_2^2} - R\right)^2 + x_3^2 = r^2$.

  We want to solve

$$(*) \quad \begin{cases} x_1 = \cos(\beta)\,(R + r\cos(\alpha)) \\ x_2 = \sin(\beta)\,(R + r\cos(\alpha)) \\ x_3 = r\sin(\alpha) \end{cases}$$

By def of cos and sin $\exists!\ \alpha, \beta \in [0, 2\pi]$ s.t.

- $\cos(\alpha) = \dfrac{\sqrt{x_1^2+x_2^2} - R}{r}$ & $\sin(\alpha) = \dfrac{x_3}{r}$ $\left(\text{indeed } \left(\frac{\sqrt{x_1^2+x_2^2} - R}{r}\right)^2 + \left(\frac{x_3}{r}\right)^2 = 1 \right)$

- $\cos(\beta) = \dfrac{x_1}{r\cos(\alpha) + R}$ & $\sin(\beta) = \dfrac{x_2}{r\cos(\alpha) + R}$ $\left(\text{indeed } (r\cos(\alpha) + R)^2 = x_1^2 + x_2^2 \right)$

Such α and β solve ⊛.

⊕ <u>Inverse</u>: From the previous computations we have that

$$\varphi^{-1}(x_1,x_2,x_3) = \begin{cases} \alpha = \arccos\left(\dfrac{\sqrt{x_1^2+x_2^2}-R}{r}\right) \\[2ex] \beta = \arccos\left(\dfrac{x_1}{\sqrt{x_1^2+x_2^2}}\right) \end{cases}$$

that defines a continuous function.

$\Rightarrow$ φ is homeom.

We also show that φ is an immersion:

B

$$D\varphi(\alpha,\beta) = \begin{pmatrix} -r\sin(\alpha)\cos(\beta) & -(R+r\cos(\alpha))\sin(\beta) \\ -r\sin(\alpha)\sin(\beta) & (R+r\cos(\alpha))\cos(\beta) \\ r\cos(\alpha) & 0 \end{pmatrix}$$

A

Note that $\det(A) = r\cos(\alpha)\cos(\beta)\,(R+r\cos(\alpha))$

$\det(B) = r\cos(\alpha)\sin(\beta)\,(R+r\cos(\alpha))$

$\Rightarrow$ if $\cos(\alpha) \neq 0$ then $\mathrm{rk}\left(D\varphi(\alpha,\beta)\right) = 2$

But if $\cos(\alpha) = 0$ then

$$D\varphi(\alpha,\beta) = \begin{pmatrix} \pm r\cos(\beta) & -R\sin(\beta) \\ \pm r\sin(\alpha) & R\cos(\beta) \\ 0 & 0 \end{pmatrix}$$

that has again rank 2.

By definition 11.1.7 we can conclude that φ is a param.

(c) By Satz 11.2.2 (4) we have

$$T_x \Upsilon = \left\{ v \in \mathbb{R}^3 : \left(\nabla f(x), v \right) = 0 \right\}$$

$$\overset{!}{=} \left\{ v \in \mathbb{R}^3 : \frac{2\left(\sqrt{x_1^2+x_2^2} - R\right)}{\sqrt{x_1^2+x_2^2}} \left(x_1 v_1 + x_2 v_2 \right) + 2x_3 v_3 = 0 \right\}$$

(d) Let

$$\begin{cases} \alpha = \arccos\left(\frac{\sqrt{x_1^2+x_2^2} - R}{r} \right) \\ \beta = \arccos\left(\frac{x_1}{\sqrt{x_1^2+x_2^2}} \right) \end{cases}$$

so that $\psi(\alpha, \beta) = (x_1, x_2, x_3)$ then by satz 11.2.2. (3) we have that

$$V_1 = \begin{pmatrix} -r \sin(\alpha) \cos(\beta) \\ -r \sin(\alpha) \sin(\beta) \\ r\cos(\alpha) \end{pmatrix} \; \& \; v_2 = \begin{pmatrix} -(R + r\cos(\alpha)) \sin(\beta) \\ (R + r\cos(\alpha)) \cos(\beta) \\ 0 \end{pmatrix}$$

satisfy $T_x \Upsilon = \operatorname{span} \langle V_1, V_2 \rangle$.

(e) By standard computations the normal vector is

$$|\nu| = \left| \frac{V_1 \times V_2}{\| V_1 \times V_2 \|} \right| = \left| C \cdot \begin{pmatrix} -\sin\alpha \cos\beta \\ -\sin\alpha \sin\beta \\ \cos\alpha \end{pmatrix} \times \begin{pmatrix} -\sin\beta \\ \cos\beta \\ 0 \end{pmatrix} \right|$$

$$= \left| C \cdot \begin{pmatrix} -\cos(\alpha)\cos(\beta) \\ -\cos(\alpha) \sin(\beta) \\ -\sin(\alpha) \end{pmatrix} \right| \quad \text{where} \quad C = \sqrt{\cos^2\alpha \cos^2\beta + \cos^2\alpha \sin^2\beta + \sin^2\alpha}$$

$$\overset{!}{=} \sqrt{\cos^2\alpha + \sin^2\alpha}$$

$$\overset{!}{=} \sqrt{1}$$

Therefore $\nu = \begin{pmatrix} \cos(\alpha)\cos(\beta) \\ \cos(\alpha)\sin(\beta) \\ \sin(\alpha) \end{pmatrix}$

Exercise 3

(a) We set $\quad \psi : [0,2\pi[\times I \longrightarrow M$

$$(\alpha,t) \longmapsto \begin{pmatrix} \cos(\alpha) f(t) \\ \sin(\alpha) f(t) \\ t \end{pmatrix}$$

then

$$D\psi(\alpha,\beta) = \begin{pmatrix} -\sin(\alpha) f(t) & \cos(\alpha) f'(t) \\ \cos(\alpha) f(t) & \sin(\alpha) f'(t) \\ 0 & 1 \end{pmatrix}$$

Note that

$$\det \begin{pmatrix} -\sin(\alpha) f(t) & \cos(\alpha) f'(t) \\ 0 & 1 \end{pmatrix} = -\sin(\alpha) f(t)$$

$$\det \begin{pmatrix} \cos(\alpha) f(t) & \sin(\alpha) f'(t) \\ 0 & 1 \end{pmatrix} = \cos(\alpha) f(t)$$

Therefore ψ is an immersion & it is also easy to see that ψ is an homeom. Hence ψ is a param. & by Satz 11.1.6 we can conclude that M is a 2-dim. manifold.

(b) Note that

$$Df(x,y) = (2x, -2y) = (0,0) \Longleftrightarrow (x,y) = (0,0)$$

Note also that $\forall\, \varepsilon > 0$ & (x_0,y_0) s.t. $(x_0,y_0) \in B(0,\varepsilon) \ni (y_0,x_0)$ then

- either $f(x_0,y_0) < 0 < f(y_0,x_0)$;
- or $f(x_0,y_0) > 0 > f(y_0,x_0)$;

and so $(x,y) = (0,0)$ is a saddle point.

We can parametrize locally around $(\frac{1}{2}, \frac{1}{4})$ the surface determined by f via the map

We can parametrize locally around $(\frac{1}{2}, \frac{1}{4})$ the surface determined by f via the map

$$\varphi : (x,y) \longmapsto (x, y, x^2 - y^2)$$

Then $D\varphi(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2x & -2y \end{pmatrix}$ & $D\varphi(\frac{1}{2}, \frac{1}{4}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -\frac{1}{2} \end{pmatrix}$.

By Satz 11.2.2. we can conclude that

$$T_{(\frac{1}{2}, \frac{1}{4})} M = \operatorname{span} \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1/2 \end{pmatrix} \right\rangle$$

Exercise 4

We want to study

$$f(x) = \int_0^\infty \underbrace{\frac{\log(1 + x^2 t^2)}{1 + t^2}}_{\varphi(x,t)} \, dt \quad , \quad x \in \mathbb{R}$$

- Differetiability? We fix $x_0 \in \mathbb{R}$ and $x_0 \neq 0$.
  - For every fixed $x \in \mathbb{R}$, $\dfrac{\log(1+x^2t^2)}{1+t^2} \in L^1((0,\infty))$. Indeed
    - $\lim_{t \to 0} \dfrac{\log(1+x^2t^2)}{1+t^2} = 0$.
    - $\dfrac{\log(1+x^2t^2)}{1+t^2} \leq C \dfrac{1}{t^{2-\delta}}$ for $\delta > 0$ & $C > 0$.
  - $\dfrac{\partial}{\partial x} \left(\dfrac{\log(1+x^2t^2)}{(1+t^2)} \right) = \dfrac{2xt^2}{(1+x^2t^2)(1+t^2)} \implies \varphi(\cdot, t)$ is differentiable.
  - I want to show that $\left| \dfrac{2t^2 x}{(1+x^2t^2)(1+t^2)} \right| \leq g(t) \in L^1((0,\infty))$ $\forall x \in (x_0 - \delta, x_0 + \delta)$

    Note that $\exists\, m, M$ s.t. $0 < m \leq |x| \leq M < \infty$ and so

$$\left|\frac{2t^2x}{(1+x^2t^2)(1+t^2)}\right| \leq \frac{\underbrace{2t^2M}{(1+m^2t^2)}}{(1+t^2)} \leq \frac{C_{M,m}}{1+t^2} \in L^1((0,\infty))$$

bounded function

By Satz 10.6.7, we conclude that f is differentiable for $x \neq 0$ and

$$f'(x) = \int_0^\infty \frac{2xt^2}{(1+t^2)(1+x^2t^2)}dt = \int_0^\infty \left(\frac{A}{1+t^2} + \frac{B}{1+x^2t^2}\right) dt =$$

$A + Ax^2t^2 + B + Bt^2 = 2xt^2$

$\begin{cases} A+B=0 \\ Ax^2+B=2x \end{cases} \Rightarrow -B=A=\frac{2x}{x^2-1}$

$$= \frac{2x}{x^2-1}\underbrace{\int_0^\infty \frac{1}{1+t^2}dt}_{=\frac{\pi}{2}} - \frac{2x}{x^2-1}\underbrace{\int_0^\infty \frac{1}{1+x^2t^2}dx}_{=\frac{\pi}{2x}}$$

$$= \frac{x\pi}{x^2-1} - \frac{\pi}{x^2-1} = \frac{(x-1)\pi}{(x^2-1)} = \frac{\pi}{x+1}$$

$\Rightarrow$ for $x \geq 0$, $f(x) = f(0) + \int_0^x \frac{\pi}{1+s}ds = \pi\left(\log(1+x)\right)$.

Then we can extend f by symmetry obtaining

$$f(x) = \pi\left(\log(1+|x|)\right).$$

Note that the function is not differentiable at zero.