# Homework VIII

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## Exercise 1

(a) Let $M \subseteq \mathbb{R}$ open interval and $f \in C^1(M, \mathbb{R})$ with $f'$ monotone increasing.

Show that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \quad \forall x,y \in M$$

Solution: W.l.o.g we assume $x < y$.

By Satz 5.2.5 (mean value theorem) $\exists \ \xi \in \left]x, \frac{x+y}{2}\right[, \ \eta \in \left]\frac{x+y}{2}, y\right[$ such that

$$f\left(\frac{x+y}{2}\right) - f(x) = f'(\xi)\left(\overbrace{\frac{x+y}{2} - x}^{\frac{y-x}{2}}\right)$$

$$f(y) - f\left(\frac{x+y}{2}\right) = f'(\eta)\left(\underbrace{y - \frac{x+y}{2}}_{\frac{y-x}{2}}\right)$$

Then we obtain that

$$\frac{f(x)+f(y)}{2} = \frac{f\left(\frac{x+y}{2}\right) - f'(\xi)\left(\frac{y-x}{2}\right) + f\left(\frac{x+y}{2}\right) + f'(\eta)\left(\frac{y-x}{2}\right)}{2}$$

$$= f\left(\frac{x+y}{2}\right) + \underbrace{\left(\frac{y-x}{4}\right)}_{\geq 0} \underbrace{\left(f'(\eta) - f'(\xi)\right)}_{\geq 0 \text{ (since } f' \text{ mon. incr.)}} \geq f\left(\frac{x+y}{2}\right).$$

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(b) Prove that

$$\left(\frac{x+y}{2}\right)^p \leq \frac{x^p + y^p}{2} \quad \forall x,y \in \mathbb{R}_{\geq 0}, \ p \in \mathbb{R}_{>1}.$$

Solution:

Use point (a) with $f(z) = z^p$ (indeed $f'(z) = p z^{p-1}$ that is monotonically increasing since $p-1 > 0$).

## Exercise 2

(a) $a,b \in \mathbb{R}$, $1 \leq p \leq q < \infty$. Show that for any $f \in L^q([a,b])$:

$$\frac{\|f\|_p}{(b-a)^{1/p}} \leq \frac{\|f\|_q}{(b-a)^{1/q}}$$

thus $\quad L^q([a,b]) \subseteq L^p([a,b])$.

**Solution:**

Let $g = |f|^p$ & $h = \chi_{[a,b]}$. Note that $\int (|f|^p)^{q/p} < \infty$ (since $f \in L^q$).

$$\|f\|_p = \left(\int_a^b |f|^p \chi_{[a,b]}\right)^{1/p} \leq \left( \left( \int_a^b (|f|^p)^{q/p} \right)^{p/q} \left( \int_a^b \chi_{[a,b]} \right)^{\frac{q-p}{q}} \right)^{1/p}$$

Holder

- $|f|^p \in L^{q/p}$
- $\frac{p}{q} + \frac{1}{x} = 1 \Rightarrow \frac{1}{x} = \frac{q-p}{q}$

$$= \left(\int_a^b |f|^q\right)^{1/q} (b-a)^{\frac{q-p}{pq}} = \|f\|_q \, (b-a)^{\frac{1}{p}-\frac{1}{q}} \qquad \square$$

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(b) $I$ unbounded interval. Show that for $1 \leq p \neq q < \infty$ both the inclusions $L^p(I) \subset L^q(I)$ and $L^q(I) \subset L^p(I)$ are false.

**Solution:**

Let $I = ]0,+\infty[$, $p=1$, $q=2$, $f: I \to \mathbb{R}$, $x \mapsto \frac{1}{x+1}$, $g: I \to \mathbb{R}$, $x \mapsto \frac{1}{\sqrt{x}} \chi_{]0,1[}(x)$

It is easy to check that $f \in L^2(I) \setminus L^1(I)$ and $g \in L^1(I) \setminus L^2(I)$.

## Exercise 3

$f,g: \mathbb{R}^n \to \mathbb{R}$ & $f \sim g \iff f-g=0$ a.e.

$\sim$ is an equiv. rel. on the space of Leb. meas. functions.

**Solution:**

$\sim$ is clearly reflexive & symmetric.

Let $f,g,h: \mathbb{R}^n \longrightarrow \mathbb{R}$ Leb. meas. fcts. s.t. $f\sim g$ & $g \sim h$.

By def. $\exists$ $A,B \subseteq \mathbb{R}^n$ s.t. $\lambda_n(A)=\lambda_n(B)=0$ s.t.

$$f=g \text{ on } A^c \quad \text{and} \quad g=h \text{ on } B^c$$

$\Longrightarrow f=h$ on $A^c \cap B^c = (A\cup B)^c$ & $\lambda_n(A\cup B)=0$

Therefore $f\sim h$. This proves transitivity. $\square$

**Exercise 4**

Let $U=(1,\infty)\times(-\pi,\pi)\times(0,\infty)$ and $\Psi: U \longrightarrow \mathbb{R}^3$ defined by

$$\Psi(r,\varphi,c)=\left(cr\cos\varphi,\ cr\sin\varphi,\ \sqrt{r^2-1}\right)$$

(a) Determine $V=\Psi(U)$ and show that $\varphi: U \longrightarrow V$ is $C^1$-diffeo.

(b) Compute $D\psi(r,\varphi,c)$

**Solution:**

**Claim:** If $\gamma=\{(x,0)\in\mathbb{R}^2 : x\leq 0\}$ then $V=\{\mathbb{R}^2\setminus\gamma\}\times(0,\infty)$

**Proof:** Easy double inclusion. (computations are similar to the ones done below)

- $\Psi$ is <u>injective</u>: Assume that $\Psi(r_1,\varphi_1,c_1)=\Psi(r_2,\varphi_2,c_2)$ then we imediatly have $r_1=r_2$ (from 3rd coordinate). We also have

$$\begin{cases} c_1\cos\varphi_1 = c_2\cos\varphi_2 & (1)\\ c_1\sin\varphi_1 = c_2\sin\varphi_2 & (2)\end{cases}$$

$\overset{(1)^2+(2)^2}{\Longrightarrow} c_1^2=c_2^2 \Longrightarrow c_1=c_2$. Finally $\varphi_1=\varphi_2$ because $\varphi\to(\cos\varphi,\sin\varphi)$ is injective in $(-\pi,\pi)$.

- $\Psi$ is <u>surjective</u>: $\forall$ $(x,y,z)\in V$ take $r=\sqrt{z^2+1}$, $c=\frac{x^2+y^2}{z^2+1}$ and

- $\Psi$ is <u>surjective</u>: $\forall\ (x,y,z) \in V$ take $r = \sqrt{z^2+1}$, $c = \frac{x^2+y^2}{z^2+1}$ and $\varphi \in (-\pi, \pi)$ with $\tan\varphi = y/x$.

- Note that

$$D_\Psi(r,\varphi,c) = \begin{pmatrix} c\cdot\cos\varphi & -cr\sin\varphi & r\cdot\cos\varphi \\ c\cdot\sin\varphi & cr\cos\varphi & r\cdot\sin\varphi \\ \frac{r}{\sqrt{r^2-1}} & 0 & 0 \end{pmatrix}$$

since all the partial derivatives are continuous $\Rightarrow \Psi$ is cont. diff.
Note also that

$$|\det D_\Psi| = c\,\frac{r^3}{\sqrt{r^2-1}} > 0$$

$\Rightarrow \Psi^{-1}$ is cont. diff. (local $\mathcal{C}^1$-diffeo but due to bij. is also a global $\mathcal{C}^1$-diffeo.)

(c) Let $H = \left\{ (x,y,z) : 0 \le z \le 2,\ \frac{1}{2}(1+z^2) \le x^2+y^2 \le 2(1+z^2) \right\}$ and compute

$$\int_H x^2 z \, d\lambda_3(x,y,z).$$

<u>Solutions:</u>

$\tilde{H} := (H \setminus \partial H) \setminus (\gamma \times \mathbb{R}) = \left\{ (x,y,z) : 0 < z < 2,\ \frac{1}{2}(1+z^2) < x^2+y^2 < 2(1+z^2) \right\}$

Since $\partial H$ & $\gamma \times \mathbb{R}$ have zero Leb. meas. and $\tilde{H}$ is open one can apply the Transformationsatz an get

$$\int_H x^2 z \, d\lambda_3 = \int_{\Psi^{-1}(\tilde{H})} (x(r,c,\varphi))^2 \, z(r,c,\varphi) \, |\det D_\Psi| \, d\lambda_3 = \circledast$$

Noting that $\Psi^{-1}(\tilde{H}) = \left\{ (r,c,\varphi) \mid 1 < r < \sqrt{5},\ -\pi < \varphi < \pi,\ \frac{1}{\sqrt{2}} < c < \sqrt{2} \right\}$

$$\circledast = \int^{\sqrt{5}} \int^{\pi} \int^{\sqrt{2}} c^2 r^2 (\cos\varphi)^2 \sqrt{r^2-1} \quad c \quad r^3 \quad dc\, d\varphi\, dr$$

$$\circledast = \int_1^{\sqrt{5}} \int_{-\pi}^{\pi} \int_{1/\sqrt{2}}^{\sqrt{2}} c^2 r^2 (\cos\varphi)^2 \sqrt{r^2-1}\; c \frac{r^3}{\sqrt{r^2-1}} \, dc\, d\varphi\, dr$$

$$= \frac{155}{8}\pi$$

## Exercise 5

Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(t,x) := \begin{cases} \frac{t^3|x|}{(t^2+x^2)^2} & \text{if } (t,x) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $\int_{\mathbb{R}} \partial_t f(t,x)\,dx = \partial_t \left( \int_{\mathbb{R}} f(t,x)\,dx \right)$.

**Solution:**

With standard computations:

$$\partial_t f(t,x) = \frac{3t^2|x|}{(t^2+x^2)^2} - \frac{4t^4|x|}{(t^2+x^2)^3}, \quad \forall x \in \mathbb{R}$$

Thus

$$\int_{\mathbb{R}} \partial_t f(t,x) = 2\int_0^{+\infty} \partial_t f(t,x) = 3t^2 \left[ -\frac{1}{(t^2+x^2)} \right]_{x=0}^{\infty} - 2t^4 \left[ -\frac{1}{(t^2+x^2)^2} \right]_{x=0}^{\infty}$$

$$= 1$$

Similarly,

$$\int_{\mathbb{R}} f(t,x) = t^3 \int_0^{+\infty} 2x\,(t^2+x^2)^{-2}\,dx = \dots = t.$$

$$\Rightarrow \partial_t \left( \int_{\mathbb{R}} f(t,x) \right) = 1$$

(b) $F(t) = \int^1 \partial_t f(t,x)\,dx \qquad G(t) = \partial_t \left( \int^1 f(t,x)\,dx \right)$

(b) $$F(t) = \int_0^1 \partial_t f(t,x)\,dx \qquad G(t) = \partial_t \left( \int_0^1 f(t,x)\,dx \right)$$

Show that $F \neq G$.

Solution:

We show that $F(0) \neq G(0)$.

Note that $\int_0^1 f(t,x)\,dx = \dots = \frac{t}{2} - \frac{t^3}{2(t^2+1)}$

and so $\partial_t \left( \int_0^1 f(t,x)\,dx \right) = \frac{1}{2} - \frac{t^4+3t^2}{2(t^2+1)^2} \Rightarrow G(0) = \frac{1}{2}$

But on the other hand $\partial_t f(t,x) = 0 \quad \forall x \in \mathbb{R}$ so $F(0) = 0$.

Motivation: $\partial_t f(t,x)\Big|_{x=t} = \frac{1}{4|t|} \xrightarrow{t \to 0} +\infty$. But $\partial_t f(0,0) = 0$.

$\Rightarrow \partial_t f$ is NOT continuous.