Homework IX

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Exercise 1

$\chi_n(x) = \chi_{[\log(n), \log(n+1)]}(x)$, $x \in \mathbb{R}$. When $\sqrt{n} \cdot \chi_n(x) \in L^p(\mathbb{R})$?

Solution:

Note that $\forall x \in \mathbb{R}$ $\sqrt{n}\, \chi_n(x) \xrightarrow{n\to\infty} 0$. Therefore we have to compute

$$\|\sqrt{n}\chi_n - 0\|_p^p = \int_{\mathbb{R}} |\sqrt{n}\chi_n(x)|^p \, dx = n^{p/2} \int_{\log(n)}^{\log(n+1)} 1 \, dx = n^{p/2} \cdot \log\left(\frac{n+1}{n}\right)$$

$$= n^{p/2} \left(\frac{1}{n} + o\left(\frac{1}{n}\right)\right) = n^{p/2 - 1} + o(n^{p/2-1}) .$$

Therefore $\|\sqrt{n}\, \chi_n\|_p \xrightarrow[n\to\infty]{} 0 \iff p/2 - 1 < 0 \iff p < 2$.

We can conclude that $(\sqrt{n}\, \chi_n)_{n\in\mathbb{N}}$ converges to 0 in $L^p(\mathbb{R})$ for all $1 \le p < 2$.

Exercise 2

$L^2(\mathbb{R}^n)$ is an Hilbert space with product $(f,g) = \int_{\mathbb{R}} f(x) g(x) \, dx$ $\forall f,g \in L^2(\mathbb{R})$.

Solution:

- $(\cdot,\cdot)$ is a SCALAR PRODUCT:
  - simmetry: $(f,g) = \int_{\mathbb{R}} fg = \int_{\mathbb{R}} gf = (g,f)$.
  - bilinearity: $(\alpha f_1 + \beta f_2, g) = \int_{\mathbb{R}} (\alpha f_1 + \beta f_2) g = \alpha \int_{\mathbb{R}} f_1 g + \beta \int f_2 g = \alpha (f_1, g) + \beta (f_2, g)$.
  - definitess: $(f,f) = 0 \iff \int_{\mathbb{R}} f^2 = 0 \underset{\text{(Sato 10.3.8)}}{\iff} f = 0$ a.e.

    Since L^p spaces are defined on the equiv. classes $f \sim g \iff f - g = 0$ a.e., we can conclude that "definitess" holds.

- positivity: $(f,f)\geq 0 \iff \int_{\mathbb{R}} |f|^2 \geq 0$ & the latter holds by Satz 10.2.14.

- $\|f\|_2 := (f,f)^{1/2} = \left(\int_{\mathbb{R}} f^2\right)^{1/2}$ is a <u>NORM</u>:

  - The fact that $\|\cdot\|_2 \geq 0$ & $\|f\|_2 = 0 \iff f=0$ follows from the computations above.
  - $\|\alpha f\|_2^2 = \left(\int_{\mathbb{R}} (\alpha f)^2\right)^{1/2} = |\alpha| \left(\int f^2\right)^{1/2} = |\alpha|\ \|f\|_2$
  - $\|f+g\|_2^2 = \int_{\mathbb{R}} (f+g)^2 = \int_{\mathbb{R}} f^2 + \int_{\mathbb{R}} g^2 + 2\int_{\mathbb{R}} fg \underset{\text{Satz 10.6.2}}{\leq} \|f\|_2^2 + \|g\|_2^2 + 2\|f\|_2\|g\|_2 = (\|f\|_2 + \|g\|_2)^2$

    $\Longrightarrow \|f+g\|_2 \leq \|f\|_2 + \|g\|_2$

- $(L^2, \|\cdot\|_2)$ is COMPLETE: This simply follows by Satz 10.3.8.

We can conclude that $(L^2, (\cdot,\cdot))$ is a Hilbert space.

Exercise 3

$f = \chi_{(0,1)}$ & $1 \leq p < \infty$.

(a) $\exists\ (f_n)_n \in T^{inc}((0,1))$ compactly supported in $(0,1)$ s.t. $\|f_n - f\|_p \longrightarrow 0$.

Solution:

Let $f_n := \chi_{[\frac{1}{n}, 1-\frac{1}{n}]}$ $\forall n \in \mathbb{N}$. Clearly, $(f_n)_n \in T^{inc}((0,1))$ and $(f_n)_n$ is compactly supported. Moreover

$$\|f_n - f\|_p^p = \int_{\mathbb{R}} \left(\chi_{(0,1)} - \chi_{[\frac{1}{n}, 1-\frac{1}{n}]}\right)^p = \left(\frac{2}{n}\right)^p \xrightarrow[n\to\infty]{} 0.$$

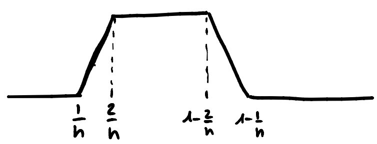
(b) $\exists\ (g_n)_n \in (\mathcal{C}((0,1)))^{\mathbb{N}}$ compactly supported in $(0,1)$ s.t. $\|f_n - f\|_p \longrightarrow 0$.

Solution:

Let

$$\begin{cases} 0 & \text{if } x < \frac{1}{n} \text{ or } x > 1-\frac{1}{n} \\ \ldots \end{cases}$$

Let
$$g_n(x) := \begin{cases} 0 & \text{if } x < \frac{1}{n} \text{ or } x > 1-\frac{1}{n} \\ nx-1 & \text{if } x \in [\frac{1}{n}, \frac{2}{n}] \\ -nx+(n-1) & \text{if } x \in [1-\frac{2}{n}, 1-\frac{1}{n}] \\ 1 & \text{if } x \in [\frac{2}{n}, 1-\frac{2}{n}] \end{cases}$$



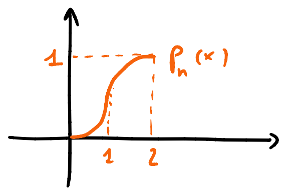
Then $g_n \in \mathcal{C}((0,1))$ and

$$\|g_n - f\|_p^p = \int_{\mathbb{R}} |\chi_{(0,1)} - g_n|^p \leq C\left(\frac{1}{n}\right)^p \longrightarrow 0$$

(c) $\exists\ (h_n)_n \in \mathcal{C}^1((0,1))$ compactly supported in $(0,1)$ s.t. $\|f_n - f\|_p \longrightarrow 0$.

Solution:

Let
$$p(x) = \frac{1}{2}\left(x^3 \chi_{(0,1)} + (2+(x-2)^3)\ \chi_{(1,2)}\right)$$



Set
$$h_n(x) = p\left(2n(x-\tfrac{1}{n})\right) \cdot \chi_{[\frac{1}{n},\frac{2}{n}]}(x) + \chi_{(\frac{1}{n}, 1-\frac{1}{n})} + p\left(2-2n(x-(1-\tfrac{2}{n}))\right)\chi_{[1-\frac{2}{n}, 1-\frac{1}{n}]}(x)$$

Since $p(x) \in \mathcal{C}^1((0,2))$ and $p'(0)=0$, $p'(2)=0$ then by construction $h_n(x) \in \mathcal{C}^1((0,1))$ with support $[\frac{1}{n}, 1-\frac{1}{n}]$.

Moreover,
$$\|h_n - f\|_p^p = \int_{\mathbb{R}} |h_n - f|^p = 2 \cdot \int_{1/n}^{2/n} \underbrace{\left|1 - p_n\left(2n\left(x-\tfrac{1}{n}\right)\right)\right|^p}_{\leq 1} dx \leq \frac{2}{n} \underset{n\to\infty}{\longrightarrow} 0.$$

Does $\lim_{n\to\infty} \|h_n' - f'\|_p = 0$ holds?

Solution:

Note that $p'(x) = \frac{1}{2}\left(3x^2\ \chi_{(0,1)} + 3(x-2)^2 \chi_{(0,2)}\right)$.

Therefore on $(\frac{1}{n}, \frac{2}{n})$:

$$\left(p\left(2n(x-\tfrac{1}{n})\right)\chi_{[\frac{1}{n},\frac{2}{n}]}(x)\right)' = 2n \cdot p'\left(2n(x-\tfrac{1}{n})\right)$$

$$\left(\rho\left(2n\left(x-\tfrac{1}{n}\right)\right)\chi_{[\frac{1}{n},\frac{2}{n}]}(x)\right)' = 2n \cdot \rho'\left(2n\left(x-\tfrac{1}{n}\right)\right)$$

In particular

$$\lim_{n\to\infty} \| h_n' - f' \|_p^p \geq n \cdot \int_{\frac{1}{n}}^{\frac{1}{n}+\frac{1}{2n}} 3\left(2n\left(x-\tfrac{1}{n}\right)\right)^2 dx \underset{\text{standard computations}}{=} \frac{1}{2}$$

thus the limit does not go to zero.

Exercise 4

$f : \mathbb{R} \to \mathbb{R}$ s.t. $g_x(t) := e^{tx} f(t) \in L^1(\mathbb{R}) \quad \forall x \in (-1,1)$.

$$\varphi(x) := \int_{\mathbb{R}} e^{tx} f(t)\, dt, \quad x \in (-1,1)$$

Show that f is differentiable.

Solution:

We want to use Lemma 10.6.7. Let $x_0 \in (-1,1)$ and note that

(i) $e^{tx} f(t) \in L^1(\mathbb{R})$, $\forall x \in (-1,1)$ by assumption

(ii) $\frac{d}{dx}\left(e^{tx} f(t)\right) = t e^{tx} f(t) \in \mathcal{C}^\infty \quad \forall t \in \mathbb{R}$.

(iii) $\forall x \in (x_0-\delta, x_0+\delta) \subseteq (x_0-2\delta, x_0+2\delta) \subseteq (0,1)$ we have

$$|t e^{tx} f(t)| = |t|\, e^{tx}\, |f(t)| \leq |t|\, e^{|t|(x_0+\delta)} |f(t)|$$
$$= \underbrace{|t|\, e^{-|t|\delta/2}}_{\leq C_\delta} e^{|t|(x_0+\frac{3}{2}\delta)} |f(t)| \leq C_\delta \cdot e^{|t|(x_0+\frac{3}{2}\delta)} |f(t)| \in L^1.$$

Therefore using lemma 10.6.7 with $]a,b[= (x_0-\delta, x_0+\delta)$ we have that φ is differentiable at x_0. Since x_0 is arbitrary, then we can conclude that φ is differentiable in $(-1,1)$.