Homework 13

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Exercise 1

Set $\varphi : U' \longrightarrow W$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ F(x_1, \dots, x_{n-1}) \end{pmatrix}$$

We have that $D\phi = \begin{pmatrix} I_{n-1} \\ \nabla F(x) \end{pmatrix}$ and so the vectors in each colum of $D\phi$ form a basis for $T_x M$.

Note that

$$(T_x M)^\perp = \operatorname{Ker} \begin{pmatrix} 1 & 0 & \cdots & 0 & \partial_{x_1} F \\ 0 & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & 0 & \\ 0 & \cdots & 0 & 1 & \partial_{x_{n-1}} F \end{pmatrix} = \left\{ v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = -v_n \nabla F(x) \right\}$$

Obviously $\nu(x) \in (T_x M)^\perp \Rightarrow \nu(x) = \begin{pmatrix} -\alpha \nabla F(x) \\ \alpha \end{pmatrix}$ with α such that $\alpha^2 \|\nabla F(x)\| + \alpha^2 = 1$

Since $\nu(x)$ points out of A, we can conclude that

$$\nu(x) = \begin{pmatrix} -\sqrt{\frac{1}{\|\nabla F(x)\|^2 + 1}} \, \nabla F(x) \\ \sqrt{\frac{1}{\|\nabla F(x)\|^2 + 1}} \end{pmatrix} \qquad \square$$

Exercise 3

(a) $D = \dfrac{Q x}{4\pi \|x\|_2^3}$, $x \neq 0$

We start by computing

$$\frac{\partial D_j}{\partial x_j} = \frac{Q}{4\pi} \frac{\partial}{\partial x_j} \left(\frac{x_j}{\|x\|_2^3} \right) = \frac{Q}{4\pi} \; \frac{\|x\|_2^3 - x_j \overbrace{\frac{\partial}{\partial x_j}\left(\|x\|_2^3\right)}^{\substack{3\|x\|_2 x_j \\ \|}}}{\|x\|_2^6}$$

$$\frac{\partial D_j}{\partial x_j} = \frac{Q}{4\pi} \frac{\partial}{\partial x_j}\left(\frac{x_j}{\|x\|_2^3}\right) = \frac{Q}{4\pi} \frac{\|x\|_2^3 - x_j \frac{\partial}{\partial x_j}(\|x\|_2^3)}{\|x\|_2^6}$$

$$= \frac{Q}{4\pi \|x\|_2^5}\left(\|x\|_2^2 - 3x_j^2\right)$$

Then we have that

$$\operatorname{div} D = \sum_{j=1}^{3} \frac{\partial D_j}{\partial x_j} = \frac{Q}{4\pi\|x\|_2^5}\left(3\|x\|_2^2 - 3\|x\|_2^2\right) = 0$$

(b) We first consider the case $0 \notin K$: Using Satz 11.3.24 we have

$$\int_{\partial_R K} \left(\frac{x}{\|x\|_2^3}, \nu\right) dS(x) = \int_K \operatorname{div}\left(\frac{x}{\|x\|_2^3}\right) = 0 \qquad (\text{by } (*))$$

Now, if $0 \in K \setminus \partial K$, we have that choosing $\varepsilon > 0$ s.t. $\overline{B(0,\varepsilon)} \subset K$:

$$\int_{\partial_R K} \left(\frac{x}{\|x\|_2^3}, \nu\right) dS(x) = \underbrace{\int_{\partial_R(K \setminus \overline{B(0,\varepsilon)})} \left(\frac{x}{\|x\|_2^3}, \nu\right) dS(x)}_{= 0 \text{ (see previous case)}} + \int_{\partial_R(\overline{B(0,\varepsilon)})} \left(\frac{x}{\|x\|_2^3}, \nu\right) dS(x)$$

Therefore it is enough to compute

$$\int_{\partial_R(\overline{B(0,\varepsilon)})} \left(\frac{x}{\|x\|_2^3}, \nu\right) dS(x) = \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\pi/2} \frac{1}{\cancel{\varepsilon^3}}\, \cancel{\varepsilon\, \varepsilon^2} \cos\beta \, d\beta \, d\alpha = 4\pi \qquad \square$$

We use the parametrization

$$\psi : (\alpha, \beta) \longmapsto \varepsilon \begin{pmatrix} \cos\alpha \cos\beta \\ \sin\alpha \cos\beta \\ \sin\beta \end{pmatrix}$$

and so $\sqrt{g_\psi(\alpha,\beta)} = \varepsilon^2 \sin\beta$ and $\nu(\psi(\alpha,\beta)) = \begin{pmatrix} \cos\alpha\cos\beta \\ \sin\alpha\cos\beta \\ \sin\beta \end{pmatrix}$

Exercise 4

We have that

$$\int_{\partial_R K} \rho\, x_3 \nu_3(x)\, dS(x) = \rho \int_{\partial_R K} (f, \nu)\, dS(x)$$

where $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix}$. Noting that $\operatorname{div} f = 1$ we can conclude (by Satz 11.3.24) that

$$\int_{\partial_R K} \rho\, x_3 \nu_3(x)\, dS(x) = \rho \int_K 1\, dx = \rho\, \lambda_3(K).$$

With similar computations we have $\int_{\partial_R K} \rho\, x_3 \nu_1(x) dS(x) = \int_{\partial_R K} \rho\, x_3 \nu_2(x) dS(x) = 0$ and therefore $F = \begin{pmatrix} 0 \\ 0 \\ \rho\, \lambda_3(K) \end{pmatrix}$.