Homework XII

lunedì 7 dicembre 2020 21:56

Exercise 2

We first note that

$$D_{\tau_a}(x) = Id_n \quad \& \quad D_\rho(x) = Q.$$

$$\Rightarrow \det\left(D_{\tau_a}(x)\right) = 1 \quad \& \quad \det\left(D_\rho(x)\right) = 1$$

Take $\xi \in \{\tau_a, \rho\}$.

Let $\{(U_i, \phi_i, V_i) \mid i \leq m\}$ be an atlas of $\mathrm{supp}(f) \cap M$ & $(\alpha_i)_{i=1}^m$ be a partition of unity on $\mathrm{supp}(f)$ s.t. $(f \circ \phi_i)\sqrt{g_{\phi_i}} \in L^1(U_i, \mathbb{R})$ $\forall i \leq m$.

Since ϕ_i is by def. both an homeom. & an immersion and

$$x \in \mathrm{supp}(f) \cap M \iff \xi(x) \in \mathrm{supp}(\xi \circ f) \cap \xi(M)$$

we have that

$$\{(U_i, \xi \circ \phi_i, \xi(V_i)) \mid i \leq m\} \text{ is an atlas of } \mathrm{supp}(\xi \circ f) \cap \xi(M)$$

Now, we have that

$$D_{\xi \circ \phi_i}(x) = D_\xi(\phi_i(x)) \cdot D_{\phi_i}(x)$$

$$\Rightarrow G_{\xi \circ \phi_i}(x) = \left(D_{\phi_i}(x)\right)^T \left(D_\xi(\phi_i(x))\right)^T D_\xi(\phi_i(x))\, D_{\phi_i}(x)$$

$$\Rightarrow g_{\xi \circ \phi_i}(x) = \det\left(D_{\phi_i}(x)^T\right) \cdot 1 \cdot 1 \cdot \det\left(D_{\phi_i}(x)\right) = g_{\phi_i}$$

Therefore $\quad (f \circ \xi^{-1}) \circ (\xi \circ \phi_i)\sqrt{g_{\xi \circ \phi_i}} = (f \circ \phi_i)\sqrt{g_{\phi_i}}$

and so $\quad f \in L^1(M, \mathbb{R}) \iff f \circ \xi^{-1} \in L^1(\xi(M), \mathbb{R}).$

In addition, we have that

$$\int_{\xi(M)} f\circ\xi^{-1} = \sum_{i=1}^{m}\int_{\mathbb{R}^k}(\alpha_i\circ\psi\circ\phi_i)(f\circ\xi^{-1}\circ\xi\circ\phi_i)\sqrt{g_{\xi\circ\phi_i}} \overset{!}{=} \int_M f$$

$\square$

Exercise 1

By definition

$$G_\phi(t) := \langle \phi'(t), \phi'(t)\rangle = \|\phi'(t)\|_2^2$$

Therefore

$$g_\phi(t) := \det(G_\phi(t)) = \|\phi'(t)\|_2^2 .$$

We can conclude that

$$\text{Length}(\phi) \underbrace{=}_{\text{Satz } 9.1.5} \int_J \|\phi'(t)\|_2\, dt = \int_{\mathbb{R}} (\chi_{Im(\phi)}\circ\phi)\sqrt{g_\phi} \underbrace{=}_{\text{Def. } 11.3.6\ (2)} \sigma_1(Im(\phi))$$

Exercise 3

Since $F: U \longrightarrow V$ is a $\mathcal{C}^k$-diffeo, $\forall\ y\in F(M)\ \exists!\ x_y\in M$ s.t.

$$F(x_y) = y.$$

Using Satz 11.1.6., $\exists$ a neighb. N_{x_y} of $x_y\in M$, an open set $A\subseteq\mathbb{R}^k$ & an immersion $\xi: A \overset{\alpha}{\hookrightarrow} N_{x_y}$.

We set $N_y := F(N_{x_y})$ & $\Psi := F \circ \xi : A \hookrightarrow N_y$

<u>Claim</u>: $\Psi : A \overset{\alpha}{\hookrightarrow} N_y$ is an immersion.

Note that if the claim is true, then by Satz 11.1.6 we can conclude that $F(M)$ is a k-dimensional manifold of class $\mathcal{C}^\alpha$.

<u>Proof of the claim</u>:

Ψ is $\mathcal{C}^\alpha$-diffeom. because composition of $\mathcal{C}^\alpha$-diffeom.

$D\psi(x) = \underbrace{DF(\xi(x))}_{\text{surjective}} \circ \underbrace{D\xi(x_0)}_{\text{surjective}}$ is surjective.

Therefore Ψ is a $\mathcal{C}^\alpha$ immersion. $\square$

Exercise 4

(a) $$\int_{\mathbb{R}^2} e^{-(x_1^2+x_2^2)}\,dx = \int_{\mathbb{R}^2} e^{-\|x\|_2^2}\,dx \underset{\text{Satz 11.3.12}}{=} \int_0^\infty \left\{ \int_{\partial B(0,r)} e^{-\overset{r^2}{\|x\|_2^2}}\, dS_x \right\} dr$$

$$= \int_0^\infty \left\{ \int_{\partial B(0,r)} e^{-r^2}\, dS_x \right\} dr = (*)$$

We can param. $\partial B(0,r)$ with $\psi : (0,2\pi) \longrightarrow \partial B(0,r)$ and so
$$t \longmapsto \begin{pmatrix} r\cdot\cos(t) \\ r\cdot\sin(t) \end{pmatrix}$$

$$D\psi(t) = \begin{pmatrix} -r\sin(t) \\ r\cos(t) \end{pmatrix} \implies g_\phi(t) = r^2(\sin^2(t)+\cos^2(t)) = r^2$$

$$(*) = \int_0^\infty \left(\int_0^{2\pi} r\cdot e^{-r^2}\, dt \right) dr = 2\pi \int^{+\infty} r e^{-r^2}\, dr = -\pi \int^\infty (-2r) e^{-r^2} dr$$

$$\circledast = \int_0^{\infty}\left(\int_0^{2\pi} r\cdot e^{-r^2}\,dt\right)dr = 2\pi\int_0^{+\infty} re^{-r^2}\,dr = -\pi\int_0^{\infty}(-2r)e^{-r^2}dr$$

$$= -\pi\left[e^{-r^2}\right]_0^{\infty} = \pi$$

(b) We start by computing the following integral:

$$\int_{\mathbb{R}_+^n} x_1^{p_1}\cdots x_n^{p_n}\underbrace{e^{-\|x\|_2^2}}_{\overset{\shortparallel}{e^{-x_1^2-\dots-x_n^2}}}dx \overset{\text{Tonelli}}{=} \prod_{i=1}^n\int_{\mathbb{R}_+} x_i^{p_i}e^{-x_i^2}\,dx_i \overset{\substack{t_i = x_i^2\\ dt_i = 2x_i\,dx_i}}{=} \frac{1}{2^n}\prod_{i=1}^n\int_{\mathbb{R}_+} t_i^{\frac{p_i-1}{2}}e^{-t_i}\,dt_i$$

$$= \frac{1}{2^n}\prod_{i=1}^n\Gamma\left(\frac{p_i+1}{2}\right).$$

Now, using Satz 11.3.12, we have that

$$\int_{\mathbb{R}_+^n} x_1^{p_1}\cdots x_n^{p_n}e^{-\|x\|_2^2}\,dx = \int_0^{\infty} r^{n-1}\left\{\int_{S_+^{n-1}}(ry_1)^{p_1}\cdots(ry_n)^{p_n}e^{-\|ry\|_2^2}S_y\right\}dr$$

$$= \underbrace{\left(\int_0^{\infty} r^{(n+p_1+\dots+p_n)-1}e^{-r^2}dr\right)}_{\overset{\text{as before}}{=}\frac{1}{2}\Gamma\left(\frac{p_1+\dots+p_n+n}{2}\right)}\left(\int_{S_+^{n-1}} y_1^{p_1}\cdots y_n^{p_n}\,dS_y\right)$$

The statement of the exercise easly follows from the results above □

Exercise 5

We already know that

- $\sigma_1(r) = 2r$
- $\sigma_2(r) = \pi r^2$ $\circledast$

We prove by induction that

$$\sigma_n(r) = \begin{cases} \dfrac{\pi^\ell}{\ell!} r^{2\ell} & n = 2\ell \\ \dfrac{2(\ell-1)!(4\pi)^{\ell-1}}{(2\ell-1)!} r^{2\ell-1} & n = 2\ell-1 \end{cases}$$

- If $\ell = 1$ the formula follows by $\circledast$.
- Inductive step: Assume the formula is true for some $\ell \geq 1$.

Let $m \in \{2(\ell+1), 2(\ell+1)-1\}$.

Note that

$$B_m(0,r) = \{x \in \mathbb{R}^m \mid \|x\|^2 < r^2\} = \{x \in \mathbb{R}^n \mid x_1^2 + \dots + x_m^2 < r^2\}$$
$$= \{x \in \mathbb{R}^m \mid x_1^2 + \dots + x_{m-2}^2 < r^2 - x_{m-1}^2 - x_m^2\}$$
$$= \{x \in \mathbb{R}^m \mid (x_1, \dots, x_{m-2}) \in B_{m-2}(0, \sqrt{r^2 - (x_{m-1}^2 + x_m^2)})\}$$

Now we take the following parametrization of $B_2(0,r)$:

$$\psi : \begin{array}{c} [0,2\pi[\times [0,r[\\ (\alpha, s) \end{array} \longmapsto \begin{pmatrix} s \cdot \cos(\alpha) \\ s \cdot \sin(\alpha) \end{pmatrix}$$

and we write $x = (\underbrace{x_1, \dots, x_{m-2}}_{y}, \underbrace{x_{m-1}, x_m}_{z}) = (y, z)$

$$\sigma_m(r) \overset{\text{Def. 11.3.6. (2)}}{=} \int_{B_m(0,r)} \chi_{B_m(0,r)} \, dx = \int_{B_2(0,r)} \int_{B_{m-2}(0,\sqrt{r^2-\|z\|^2})} \chi_{B_{m-2}(0,\sqrt{r^2-\|z\|^2})} \, dy \, dz$$

$$= \int_0^{2\pi}\int_0^r \underbrace{\left(\int_{B_{m-2}(0,\sqrt{r^2-s^2})} \chi_{B_{m-2}(0,\sqrt{r^2-s^2})}\,dy\right)}_{=\sigma_{m-2}(\sqrt{r^2-s^2})} \underbrace{|\det D\varphi|}_{= s \quad \text{(as in Ex. 4)}}\,ds\,d\alpha$$

$$= 2\pi \int_0^r \underbrace{\sigma_{m-2}(\sqrt{r^2-s^2})}_{(\sqrt{r^2-s^2})^{m-2}\,\sigma_{m-2}(1)} \cdot s\,ds = 2\pi\,\sigma_{m-2}(1)\cdot\left(-\frac{1}{2}\right)\int_0^r (r^2-s^2)^{\frac{m-2}{2}}(-2s)\,ds$$

$$= -\pi\,\sigma_{m-2}(1)\cdot\left[\rho^{\frac{m}{2}}\,\frac{2}{m}\right]_{r^2}^{0} = 2\pi\,\sigma_{m-2}(1)\,r^m/m$$

$\rho = r^2-s^2$

$d\rho = -2s\,ds$

$$\Rightarrow \sigma_m(r) = \frac{2\pi r^m}{m}\,\sigma_{m-2}(1) \qquad (*)$$

Now by inductive hp we have that

$$\sigma_{m-2}(1) = \begin{cases} \dfrac{\pi^{\ell}}{\ell!} & \text{if } m = 2(\ell+1) \Rightarrow m-2 = 2\ell \\[2ex] \dfrac{2\,(\ell-1)!\,(4\pi)^{\ell-1}}{(2\ell-1)!} & \text{if } m = 2(\ell+1)-1 \Rightarrow m-2 = 2\ell-1 \end{cases}$$

Therefore by $(*)$ we obtain that

$$\sigma_m(r) = \begin{cases} \dfrac{2\pi r^{2\ell+2}}{2(\ell+1)}\,\dfrac{\pi^{\ell}}{\ell!} = \dfrac{\pi^{\ell+1}\,r^{2(\ell+1)}}{(\ell+1)!} & \text{if } m = 2(\ell+1) \\[2ex] \dfrac{2\pi r^{2\ell+1}}{2\ell+1}\,\dfrac{2\,(\ell-1)!\,(4\pi)^{\ell-1}}{(2\ell-1)!} = \dfrac{2\,\ell!\,(4\pi)^{\ell}\,r^{2\ell+1}}{(2\ell+1)!} & \text{if } m = 2(\ell+1)-1 \end{cases}$$

$\square$