Homework X

martedì 24 novembre 2020 16:30

Exercise 5

Note that

$$\gamma_1'(t) = \begin{pmatrix} 2\left(\cos^2(t) - \sin^2(t)\right) \\ -4\cos(t)\sin(t) \\ 2\cos(t) \end{pmatrix}$$

$$\gamma_2'(t) = \begin{pmatrix} 2t\cos(t) - t^2\sin(t) \\ 2t\sin(t) + t^2\cos(t) \\ t \end{pmatrix}$$

Therefore

$$\|\gamma_1'\| = \sqrt{4\left(\cos^4(t) + \sin^4(t) - 2\cos^2(t)\sin^2(t)\right) + 16\cos^2(t)\sin^2(t) + 4\cos^2(t)}$$

$$= \sqrt{4\cos^4(t) + 4\sin^4(t) + 8\sin^2(t)\cos^2(t) + 4\cos^2(t)}$$

$\sin^2(t) = 1-\cos^2(t)$

$$= \sqrt{4\cos^4(t) + 4\left(1-\cos^2(t)\right)^2 + 8\left(1-\cos^2(t)\right)\cos^2(t) + 4\cos^2(t)}$$

$$= \sqrt{\cancel{4\cos^4(t)} + 4 + \cancel{4\cos^4(t)} - \cancel{8\cos^2(t)} + \cancel{8\cos^2(t)} - \cancel{8\cos^4(t)} + 4\cos^2(t)}$$

$$= 2\sqrt{\cos^2(t)+1}$$

and

$$\|\gamma_2'\| = \sqrt{4t^2\cos^2(t) + t^4\sin^2(t) - \cancel{4t^3\cos(t)\sin(t)} + 4t^2\sin^2(t) + t^4\cos^2(t) + \cancel{4t^3\sin(t)\cos(t)} + t^2}$$

$$= \sqrt{4t^2 + t^4 + t^2} = t\sqrt{5+t^2}$$

We can now compute the integrals:

- $$\int_{\gamma_1} z \, ds = 4\int_0^{\pi} \sin(t)\sqrt{\cos^2(t)+1}\, dt = 4\int_{-1}^{1} \sqrt{u^2+1}\, du$$

  $u = \cos(t)$

  $du = -\sin(t)\, dt$

$$= 4\left[\frac{1}{2}\left(u\sqrt{u^2+1} + \sinh^{-1}(u)\right)\right]_{u=-1}^{1} = 4\left[\frac{1}{2}\left(\sqrt{2}+\sinh^{-1}(1)\right) - \frac{1}{2}\left(-\sqrt{2}+\sinh^{-1}(-1)\right)\right]$$

$\sinh^{-1}(x) = \log\left(\sqrt{x^2+1}+x\right)$

$$= 4\left(\sqrt{2} + \log\left(1+\sqrt{2}\right)\right)$$

- $$\int_{\gamma_2} x^2+y^2\, ds = \int_0^{\pi} t^5\sqrt{5+t^2}\, dt = \left[\frac{1}{21}\left(t^2+5\right)^{3/2}\left(3t^4-12t^2+40\right)\right]_{t=0}^{\pi}$$

$$= \frac{1}{21}\left(\pi^2+5\right)^{3/2}\left(3\pi^4-12\pi^2+40\right) - \frac{5^{3/2}}{21}\cdot 40$$

Exercise 1

($\Rightarrow$) Assume that $f \in L^p(A)$ then

$$\infty > \int_A |f(x)|^p\, dx = \sum_{k=1}^{\infty}\int_{A_k}|f(x)|^p\, dx \geq \sum_{k=1}^{\infty}\int_{A_k}(k-1)^p\, dx =$$

$$= \sum_{k=1}^{\infty}(k-1)^p \lambda_n(A_k).$$

Therefore $\sum_{k=1}^{\infty}(k-1)^p\lambda_n(A_k) < \infty$. Since

$$\sum_{k=1}^{\infty}k^p\lambda_n(A_k) \underset{k\leq 2(k-1)\ \forall k\geq 2}{\leq} \sum_{k=1}^{\infty}\left(2(k-1)\right)^p\lambda_n(A_k) + \lambda_n(A_1) = 2^p\underbrace{\sum_{k=1}^{\infty}(k-1)^p\lambda_n(A_k)}_{<\infty} + \underbrace{\lambda_n(A_1)}_{<\infty} < \infty$$

we can conclude that $\sum_{k=1}^{\infty}k^p\lambda_n(A_k) < \infty$. $\square$

($\Leftarrow$) Assume that $\sum_{k=1}^{\infty}k^p\lambda_n(A_k) < \infty$ then

$$\int_A |f(x)|^p\, dx = \sum_{k=1}^{\infty}\int_{A_k}|f(x)|^p\, dx \leq \sum_{k=1}^{\infty}k^p\lambda_n(A_k) < \infty.$$

Since f is meas. by assumption $f \in L^p(A)$. $\square$

Exercise 2

We set $M_i = \{ (x,y) \in \mathbb{R}^2 \mid f_i(x,y) = 0 \}$ $\forall$ $i \in \{1,2,3,4,5\}$.

- $f_1(x,y) = y^2 - x^3$

  M_1 is <u>not</u> a manifold indeed if $(x_0,y_0) = (0,0) \in M_1$ then

  $$\nabla f_1(x_0,y_0) = \begin{pmatrix} -3x_0^2 \\ 2y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- $f_2(x,y) = x^2 + y^2 - 1$

  M_2 is a manifold indeed $f_2 \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$ and

  $$\nabla f_2(x,y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \forall \; (x,y) \in M_2$$

- $f_3(x,y) = y^2 - x^2(x+1)$

  M_3 is <u>not</u> a manifold indeed if $(x_0,y_0) = (0,0) \in M_3$ then

  $$\nabla f_3(x_0,y_0) = \begin{pmatrix} -2x_0(x_0+1) - x_0^2 \\ 2y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- $f_4(x,y) = y^2 - x^2$

  M_4 is <u>not</u> a manifold, indeed if $(x_0,y_0) = (0,0) \in M_4$ then

$$\nabla f_4(x_0,y_0) = \begin{pmatrix} -2x_0 \\ 2y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- $f_5(x,y) = x^2 - 4y^2(1-y^2)$

  M_5 is <u>not</u> a manifold, indeed if $(x_0,y_0) = (0,0) \in M_5$ then

$$\nabla f_5(x_0,y_0) = \begin{pmatrix} 2x_0 \\ -8y_0(1-y_0^2) + 8y_0^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Exercise 3

(a) Define $g(x,y) = \begin{pmatrix} x^3 \\ y \end{pmatrix}$ then $D_g(x,y) = \begin{pmatrix} 3x & 0 \\ 0 & 1 \end{pmatrix}$ and so $g \in \mathcal{C}^1(\mathbb{R}^2,\mathbb{R}^2)$. Clearly g is a bijective function and its inverse is

$$g^{-1}(x,y) = \begin{pmatrix} \sqrt[3]{x} \\ y \end{pmatrix}$$

and g^{-1} is therefore not differentiable. $\square$

(b) By proposition 8.2.13, $f \in \mathcal{C}^1(V', \mathbb{R}^3)$ where V' is a neighbourhood of p_0.
By proposition 8.5.7,
- $\exists$ V'' neighb. of p_0, U neighb. of $f(p_0)$ s.t. $f : V'' \longrightarrow U$ is bijective
- $f|_{V''}^{-1} = h \in \mathcal{C}^1(U, V'')$.

Again by proposition 8.2.13, h has continuous partial derivatives.
Clearly, $\forall\, y \in U$, $f(h(y)) = y$. $\square$

Exercise 4

- $\phi : \overbrace{]t_0-\frac{\pi}{2}, t_0+\frac{\pi}{2}[\times]-1,1[}^{:=D} \longrightarrow \phi(D)$ is a homeomorphism $\forall\, t_0 \in \mathbb{R}$

We fix $t_0 = 0$ (If $t_0 \neq 0$ it is enough to use the periodicity of cos & sin).

⊕ Injectivity: Follows using the same computations used for showing that there exists an inverse map (see below)

⊕ Surjectivity: Clear by definition.

⊕ Continuity of the inverse map:

$$\begin{cases} \cos(2t)\,(2-r\sin(t)) = y_1 & (1) \\ \sin(2t)\,(2-r\sin(t)) = y_2 & (2) \\ r\cos(t) = y_3 & (3) \end{cases}$$

Assume now that $\boxed{y_1 \neq 0}$: Then from (3) : $\cos(t) \neq 0$ (since $t \in]-\frac{\pi}{2}, \frac{\pi}{2}[$) & $r = \frac{y_3}{\cos(t)}$

Since $\cos(2t) \neq 0$ (because $y_1 \neq 0$) then : From (1) : $2 - r\sin(t) = \frac{y_1}{\cos(2t)}$ since $\cos(2t) \neq 0$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Substituting in (2) $\tan(2t)\, y_1 = y_2 \implies \tan(2t) = y_2/y_1$

Therefore : $t = \frac{1}{2}\arctan\left(\frac{y_2}{y_1}\right)$ $\quad r = \frac{y_3}{\cos(t)} = \frac{y_3}{\cos(\arctan(y_2/y_3))}$

This map is well-defined as soon as $y_1 \neq 0$.

- If $y_1 = 0$ then

$$\begin{cases} \cos(2t)\,(2-r\sin(t)) = 0 \\ \sin(2t)\,(2-r\sin(t)) = y_2 \\ y_3 = r\cos(t) \end{cases}$$

Since $(2-r\sin(t))>0$ then $\cos(2t)$ must be zero.

So t is either $\frac{\pi}{4}$ or $-\frac{\pi}{4}$

- If $t=\frac{\pi}{4}$ we obtain that $y_2>0$ and

$$\begin{cases} 2-r\frac{1}{\sqrt{2}}=y_2 \\ r=y_3\sqrt{2} \end{cases} \Rightarrow \begin{cases} r=(2-y_2)\sqrt{2} \\ r=y_3\sqrt{2} \end{cases} \Rightarrow \boxed{y_2=2-y_3}$$

- If $t=-\frac{\pi}{4}$ we obtain that $y_2<0$

$$\begin{cases} -\left(2+r\frac{1}{\sqrt{2}}\right)=y_2 \\ r=y_3\sqrt{2} \end{cases} \Rightarrow \begin{cases} r=(-2-y_2)\sqrt{2} \\ r=y_3\sqrt{2} \end{cases} \Rightarrow y_2=-2-y_3$$

We conclude that:

$$\phi^{-1}(y_1,y_2,y_3)=\begin{cases} t=\frac{1}{2}\arctan(y_2/y_1) & r=\dfrac{y_3}{\cos(t)} & \text{if } y_1\neq 0 \\ t=\frac{\pi}{4} & r=y_3/\cos(t) & \text{if } y_1=0 \ \& \ y_2=2-y_3>0 \\ t=-\frac{\pi}{4} & r=y_3/\cos(t) & \text{if } y_1=0 \ \& \ y_2=-2-y_3<0 \end{cases}$$

Noting that:
- if $y_2>0$ then $\lim_{y_1\to 0}\frac{1}{2}\arctan(y_2/y_1)=\frac{\pi}{4}$
- if $y_2<0$ then $\lim_{y_1\to 0}\frac{1}{2}\arctan\left(y_2/y_1\right)=-\frac{\pi}{4}$
- if $y_1=0$ then $y_2\neq 0$ in $\phi(D)$

We conclude that $\phi^{-1}(y_1,y_2,y_3)$ is continuous on $\phi(D)$.

This shows that $\phi: D\longrightarrow\phi(D)$ is homeomorphism.

Now note that

A

$$D\phi\,(t,r) = \begin{pmatrix} 2\sin(2t)\,(r\sin(t)-2) - r\cos(t)\cos(2t) & \sin(t)\,(-\cos(2t)) \\ \cos(2t)\,(4-2r\sin(t)) - r\sin(2t)\cos(t) & -\sin(t)\sin(2t) \\ -r\sin(t) & \cos(t) \end{pmatrix}$$

and

$$\det(A) = \ldots\ldots = 4\sin(t) - 2r\sin^2(t) \neq 0 \quad \text{as soon as} \quad t \neq 0$$

but note that if $t=0$ then

$$D\phi\,(0,r) = \begin{pmatrix} -r & 0 \\ 4 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore $Rk(D\phi\,(t,r)) = 2 \quad \forall\ (t,r)$ and therefore ϕ is an immersion.