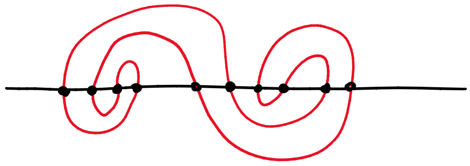
# Meanders & Meandric systems

## 1. Introduction

Henry Poincaré, 1912:

"In how many ways a simple loop in the plane can cross a line at a specified number of points?"



self-avoiding simple curve

which we always fix to be $\{1, \dots, 2n\} \subseteq \mathbb{Z}$

**Definition:** A meander of size $n$ is a simple loop which crosses a strait line at $2n$ points.
A configuration is defined up to homeomorphisms of the plane which fix the line and map the upper-half plane into the upper-half plane.

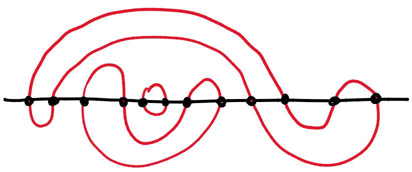
Let

$$M_n := \# \text{ of meanders of size } n$$

then the previous question can be rephrased as:

Can we determine $M_n$?

**Definition:** A meandric system of size $n$ is a set of disjoint simple loops in the plane which cross a strait line at $2n$ points (+ same equiv. relation as above).



**Remark:** A meander is just a meandric system with one loop.

The number of meandric systems $m_n$ of size $n$ is <u>much</u> simpler to compute:

Note that the parts of the loops above (resp. below) the line form an arc diagram, i.e. a non-crossing perfect matching of the points $\{1, \dots, 2n\}$, and so

$$m_n = C_n^2 \qquad \text{where} \qquad C_n = \frac{1}{n+1}\binom{2n}{n} = n\text{-th Catalan number.}$$

In particular,

constant exponential polynomial

$$m_n \sim \left(\frac{4^n}{n^{3/2}\sqrt{\pi}}\right)^2 = 1/\pi \cdot 16^n \cdot n^{-3}$$

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What about $M_n$? Obviously,

$$C_n \leq M_n \leq C_n^2$$

Any top arc-diagram can be completed in at least one way into a meander. (Exercise)

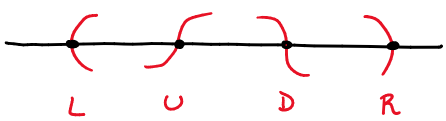
Beautiful survey paper on meanders / meandric systems

It is conjecture (in various papers; see "Meanders: A personal perspective" by Alexander Zvonkin) that

$$M_n \sim C \cdot A^n \cdot n^{-\alpha} \quad \text{for some } C, A, \alpha > 0.$$

(Same "structure" as meandric systems & many others comb. structures...)

Encoding every meander as a sequence in $\{L, U, D, R\}^{2n}$ by assigning:




and imposing that $M_n$ must avoid "LR" one gets:

$$A \leq (2+\sqrt{3})^2 \approx 13.92.$$

And with a more sophisticated analysis, Albert & Paterson, 2005, obtained that

$$11.380 \leq A \leq 12.901$$

Numerical simulations (by Jensen & Guttmann) suggests that

$$A \approx 12.26287$$

To the best of my knowledge, there is no conjecture for the exact value of A or even for its nature.

Considering the generating function

$$M(t) = \sum_{n=0}^{\infty} M_n \cdot t^{2n}$$

and noting that $M_{a+b} \geq M_a \cdot M_b$, we get that

$$A := \lim_{n \to \infty} M_n^{1/n} \quad \text{exists}$$

and $A = \left(\frac{1}{\rho}\right)^2$ where $\rho$ is the radius of convergence of $M(t)$.

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To the best of my knowledge, this is all we rigourously know about the Poincaré's questions after more than 100 years....

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But the story becomes even more interesting around 2000, when a remarkable physics paper by Di Francesco, Golinelli, and Guitter conjectured (again) that

$$M_n \sim C \cdot A^n \cdot n^{-\alpha}$$

but they additionally conjectured the exact value for $\alpha$:

$$\alpha = \frac{29 + \sqrt{145}}{12} \approx 3.420132882$$

Numerical simulations (again by Jensen & Guttmann) confirms this prediction (Simulations for the exponent $\alpha$ match with the conjectured value up to three digits after the comma).

A natural question is:

"Why the conjecture is for $\alpha$ not for $A$?"

In some sense, $\alpha$ (sometime called the "CRITICAL EXPONENT") is more important than $A$:

- In physics, $\alpha$ controls the type of phase transition.
- In combinatorics, $\alpha$ controls the type of singularity of $M(t)$ at the radius of convergence.

Di Francesco, Golinelli, and Guitter arrived to the above conjecture combing two ingredients:

1. Describe the 'central charge $c$' of a limiting Conformal Field Theory for meanders ($c=-4$)
2. Use the KPZ-equation (Knizhnik-Polyakov-Zamolodchikov) to derive $\alpha$ from $c$.

Motivations: On top of being simple & beautiful comb. objects, meanders & meandric systems have been used in physics as models for polymer folding.
Moreover these objects are related to many areas of mathematics:

- COMBINATORICS
- PROBABILITY
- GEOMETRY OF MODULI SPACES OF SQUARED TILED SURFACES (Delecroix, Goujard, Zograf, Zorich, 2019)
- COMPLEX ANALYSIS
- ...

## 2. The plan of this mini-course

In this mini course we will mainly focus on describing <u>the geometry</u> of uniformly random meanders & meandric systems:

- In the <u>1st part</u> we will focus on <u>meandric systems</u>:
  1. We will first present a new conjecture describing <u>the limiting geometry for meandric systems</u>.
  2. We will present <u>several rigourous results</u> which are consistent with the conjecture above.

- In the <u>2nd part</u> we will focus on <u>meanders</u>:
  1. We will propose (a surprising) limiting object for meanders, called <u>THE MEANDRIC PERMUTON</u>
  2. We will prove some <u>interesting properties</u> of this new object.

If I have some time at the end, I also plan to explain a bit more the connections between the physics conjectures mentioned above and our conjectures. The curious reader can have a look at:

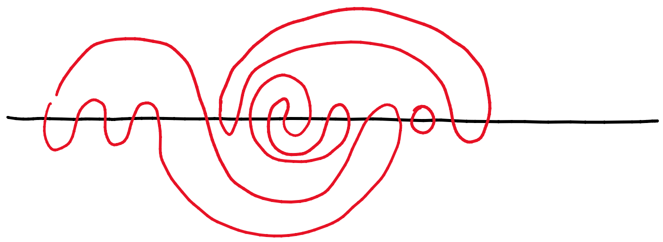
× "Permutons, meanders, & SLE-decorated Liouville quantum gravity". Borga, Gwynne, Sun.
  ↳ Section 6

× "On the geometry of uniform meandric systems". Borga, Gwynne, Park.
  ↳ Section 7

# 3. On the geometry of uniform meandric systems

## 3.1 Previous works on meandric systems

Recall that a meandric system is the following combinatorial structure:



Some basic questions about a uniform meandric system $m_n$ of size $n$:

① How can I sample a uniform meandric system $m_n$?

② How many loops in $m_n$?

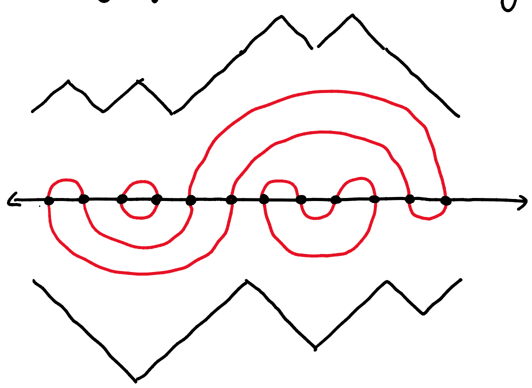
③ What is the size of the largest loop in $m_n$?

④ Is there typically a single loop of $m_n$ which is much larger than the other loops? Or, are there multiple loops of comparable size?

⑤ Is there a sort of scaling limit for $m_n$?

① How can I sample a uniform meandric system $m_n$?

This question has a (surprisingly) simple answer (note that this is not the case for meanders). Indeed, we have the following bijection between meandric systems & pairs of walk excursions:



Then, in order to sample a uniform meandric system of size $n$, it is enough to sample an <u>independent</u> pair of walk excursions of time duration $2n$ and then apply the bijection above.

One might now claim that since every question on the meandric system can be rephrased as a question on an independent pair of walk excursions of time duration $2n$, then most of the questions above should be "easy".

We will see that this is <u>NOT</u> the case! Why? Loops are a <u>very</u> complicated functional of the two walks, which is quite hard to study.

## ② How many loops in $m_n$?

The result was conjectured by Golden, Nica, Puder, 2020

This question has been solved by Féray & Thévenin (2022):

<u>Theorem</u>: There exists a constant $c \in (0,1)$ such that

$$\frac{\#\text{ loops in } m_n}{n} \xrightarrow[n\to\infty]{\mathbb{P}} c$$

Moreover, $0.207 \leq c \leq 0.292$. [Numerical symulations: 0,23]

The key lemma for the proof is the following one:

<u>Lemma</u>: Let $i_n$ be a uniform number in $\{1, \dots, 2n\}$, then

$$\frac{\#\text{ loops in } m_n}{n} = \mathbb{E}\left[\frac{2}{|\ell_{i_n}(m_n)|} \,\middle|\, m_n\right],$$

where $|\ell_{i_n}(m_n)|$ denotes the size of the loop in $m_n$ containing $i_n$.

Proof:

$$\mathbb{E}\left[\frac{2}{|\ell_{i_n}(m_n)|} \,\middle|\, m_n\right] = \sum_{i=1}^{2n} \frac{1}{\cancel{2}n} \underbrace{\frac{\cancel{2}}{\ell_{i_n}(m_n)}}_{} = \frac{\#\text{ loops in } m_n}{n} \qquad \square$$

Each loop $\ell$ in $m_n$ contributes $\frac{1}{|\ell|} \times |\ell| = 1$ to the sum.

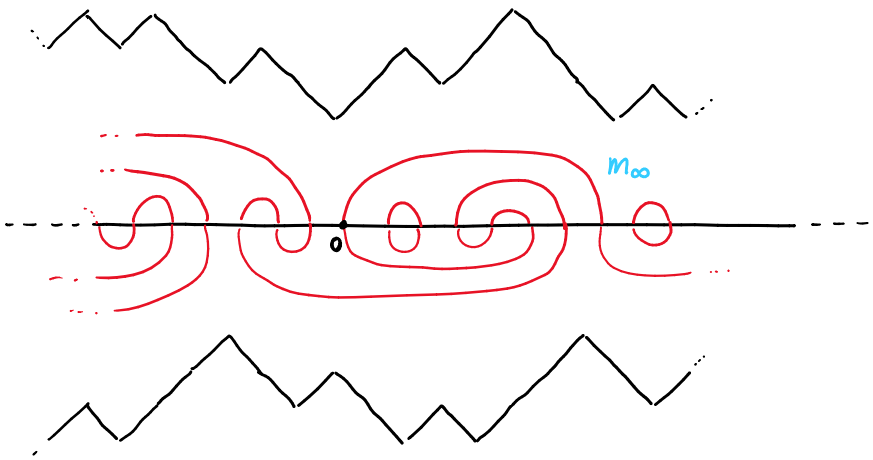
The rest of the proof of the theorem amounts in proving that

$$\mathbb{E}\left[\frac{2}{\ell_{in}(m_n)} \,\middle|\, m_n\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[\frac{2}{|\ell_0(m_\infty)|}\right] =: c$$

where $m_\infty$ is the (quenched) Benjamini-Schramm local limit of $m_n$.

In particular, $m_\infty$ is shown to be the uniform infinite meandric system (UIMS) $m_\infty$ introduced by Curien, Kozma, Sidoravicius, Tournier. (2017) ⤳ We will come back to this paper

$m_\infty$ is just the infinite meandric system obtained by replacing the two walk excursions above with two bi-infinite simple random walks:




Note that, in particular, we get that

$$c = \mathbb{E}\left[\frac{2}{|\ell_0(m_\infty)|}\right] = \sum_{n=1}^{\infty} \frac{\cancel{2}}{\cancel{2}n} \sum_{m\in\mathcal{M}_n} \overbrace{\mathbb{P}\left(\mathrm{Shape}(\ell_0(m_\infty))=m\right)}^{:=p_m} .$$

$\mathcal{M}_n$: meanders of size $n$

Moreover, these probabilities $p_m$ can be explicitly computed in certain simple cases:

$$p_{\ominus} = \frac{2}{\pi} - \frac{1}{2} \approx 0.137$$

$$p = \frac{1}{4} - \frac{2}{3\pi} \approx 0.038$$

Open question: Is it true that $p_m \in \mathbb{Q}\left[\frac{1}{\pi}\right]$, $\forall m \in \mathcal{M}$ ?

Update (04/10/2024) The question has been solved by Boston, Ferry, Thevenin.

③ What is the size of the largest loop?

Kargin (2020) shows that $\exists\, c>0$ s.t. w.h.p. (i.e. with probability one when $n \to \infty$):

$$\text{Size of the largest loop} \geq c \cdot \log(n).$$

His numerical simulations suggests that

$$\text{Size of the largest loop} \approx n^{\alpha} \quad \text{with} \quad \alpha = 4/5.$$

(Our simulations gives $\alpha \approx 0.792\ldots$)

④ Is there typically a single loop of $m_n$ which is much larger than the other loops? Or, are there multiple loops of comparable size?

This question is closely related to the following question asked by Curien, Kozma, Sidoravicius, Tournier: (2017)

Is there an infinite component in $m_\infty$? (They call it the "INFINITE NOODLE")

Theorem:

$$\mathbb{P}(\#\text{ of infinite components of } m_\infty \in \{0,1\}) = 1 \quad \& \quad \mathbb{P}(\text{There is no INFINITE NOODLE}) \in \{0,1\}.$$

Conj: $\mathbb{P}(\text{There is no INFINITE NOODLE}) = 1.$ [But without any specific motivation]

⑤ Is there a sort of scaling limit for $m_n$?

To the best of my knowledge nothing is known/conjectured before our work

## 3.3 A conjectural scaling limit

We can look at a meandric system $m_n$ as a triplet

$$(M_n, P_n, \Gamma_n)$$

$M_n$ → **PLANAR MAP:**

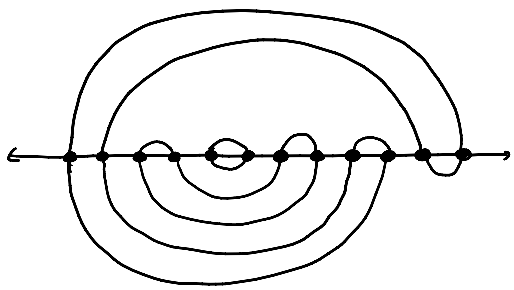
VERTICES = $\{1, \dots, 2n\}$

EDGES = { arcs of meandric systems + pieces of the line between consecutive vertices + one additional edge from $2n$ to $1$ }

$P_n$ → HAMILTONIAN CYCLE — determined by the line

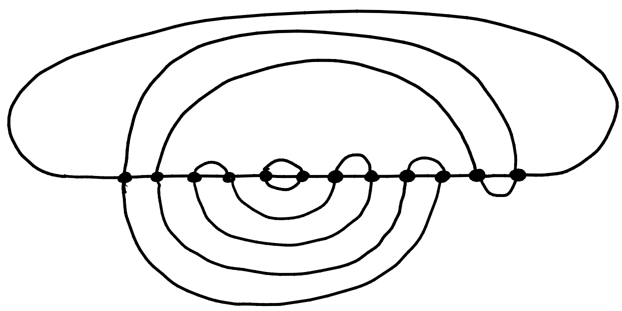
$\Gamma_n$ → COLLECTION OF LOOPS — determined by the arcs

Example: If we consider this meandric system:

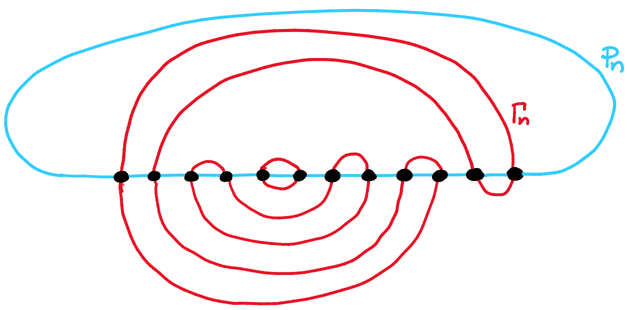


then

$M_n$ =



and



**Conjecture:** (B., Gwynne, Park '23)

$(M_n, P_n, \Pi_n)$ converges under an appropriate scaling limit to an independent triplet consisting of

$$(\sqrt{2}\text{-LQG sphere},\ SLE_8,\ CLE_6)$$

**What are these objects?**

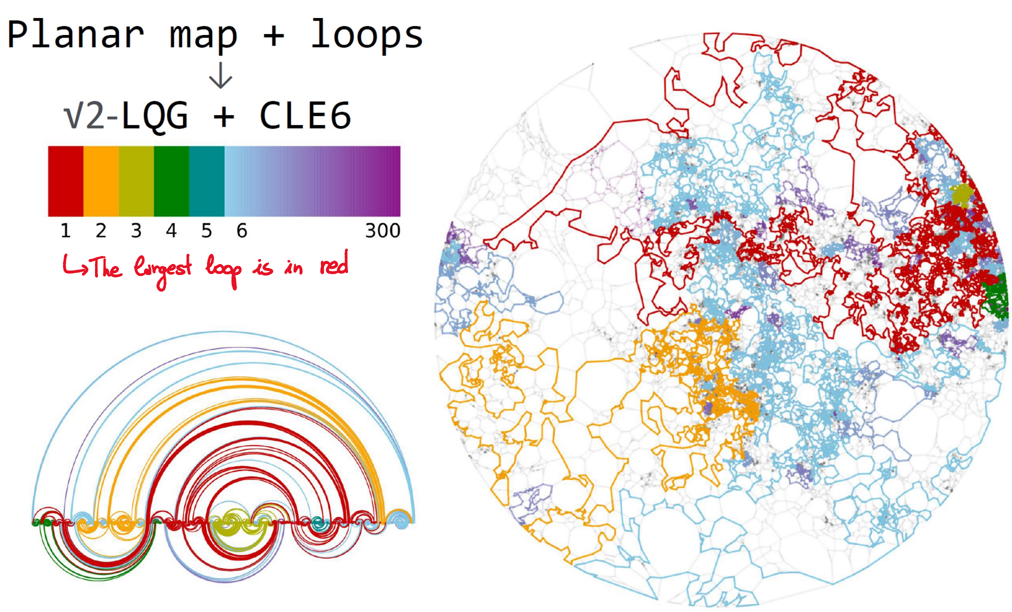
- The Liouville quantum gravity ($\gamma$-LQG) sphere with parameter $\gamma \in (0,2]$ is a random fractal surface with the topology of the sphere (it can be described by a random metric and a random measure on the Riemann sphere $\mathbb{C} \cup \{\infty\}$).

- The whole-plane Schramm-Loewner evolution ($SLE_\kappa$) with parameter $\kappa \geq 8$ is a space-filling (but non self-crossing) random fractal curve on $\mathbb{C} \cup \{\infty\}$.

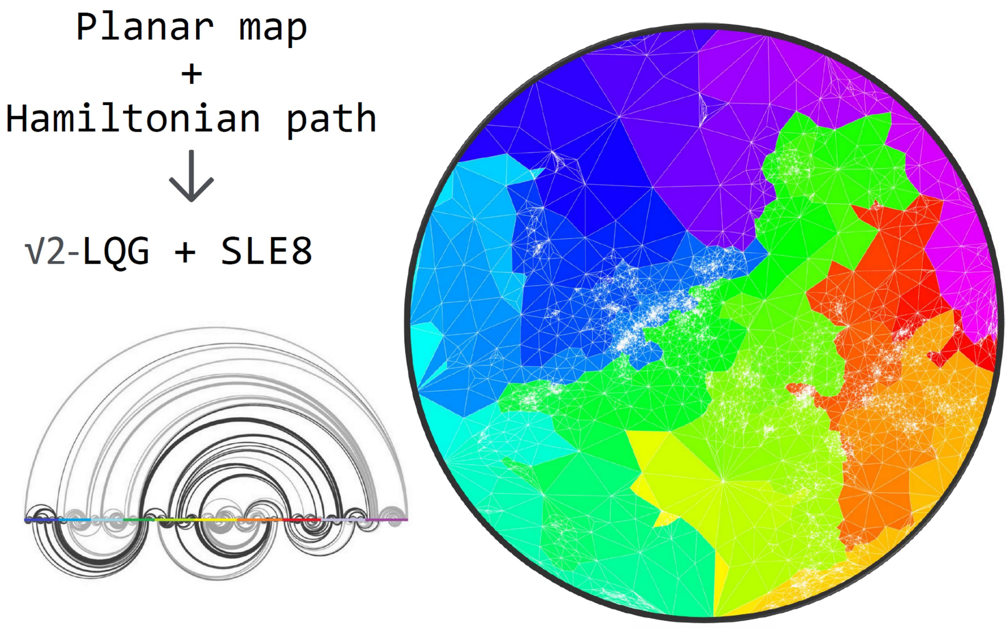
- The whole-plane conformal loop ensemble ($CLE_\kappa$) with $\kappa \in (8/3, 8)$ is a random collection of loops which do not cross themselves or each other and wich locally look like $SLE_\kappa$.

**Some simulations** (which are important to get the correct intuition):

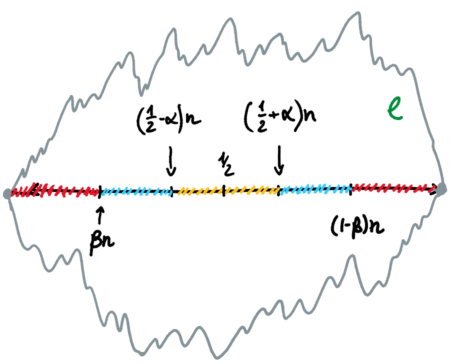
Here we embedded the meandric sistem $m_n$ in the sphere using the Tutte embedding: every vertex is the baricenter of the neighboors



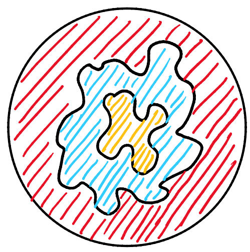
Here we embedded another meandric system $m_n$ in the sphere and we color each face of the embedded dual map using the same color of the corresponding vertex in $m_n$ (see the left-hand side):




One important fact is the following one: When we embed the meandric system we can find $\alpha > 0$ and $\beta > 0$ such that:




After embedding in the sphere we see the following in the limit:



All the 3 regions (red, blue, yellow) are macroscopic & red and yellow are separated by blue.

Some remarks:

- The conjecture is based on a combination of rigorous results, physics heuristics, and numerical simulations on a much more general model called

$$O(n \times m)\text{-loop model on planar maps.}$$

See

× "Permutons, meanders, & SLE-decorated Liouville quantum gravity". Borga, Gwynne, Sun.
↳ Section 6

× "On the geometry of uniform meandric systems". Borga, Gwynne, Park.
↳ Section 7

for more explanations.

- The type of convergence should be quite general: for instance w.r.t. Gromov-Hausdorff topology for metric spaces or w.r.t. some (nice) embedding of $M_n$ into $\mathbb{C} \cup \{\infty\}$.

- The 3 limiting objects are all independent (this is NOT true at the discrete level).
- ($\sqrt{2}$-LQG, $SLE_8$) is the same limit as uniform planar maps decorated by a spanning tree.
- $CLE_6$ is the limit of the boundaries of clusters in Bernoulli percolation on the exagonal lattice
- We know how to prove that $(M_n, P_n) \longrightarrow (\sqrt{2}\text{-LQG}, SLE_8)$ at least in the Peano-sphere sense [this is a weak type of convergence but it is VERY HELPFULL for our results, see later...]
- The really hard part of the conjecture is to prove that $\Gamma_n \longrightarrow CLE_6$.

  One aspect that might explain why this is hard is the following fact:
  If $(e^+, e^-)$ are the two limiting Brownian excursions for the pair of walk excursions determining $m_n$, then

  $$(e^+, e^-) \quad \text{and} \quad (\sqrt{2}\text{-LQG}, SLE_8)$$

  determines each other (in the sense that one is a meas. function of the other one) but $CLE_6$ is not determined by $(e^+, e^-)$.
- This conjecture gives several new (conjectural) answers to the questions above. In particular:

3. What is the size of the largest loop?
4. Is there typically a single loop of $m_n$ which is much larger that the other loops? Or, are there multiple loops of comparable size?

Conjecture:

$$\#\{\text{vertices of the } k\text{-th largest loop of } m_n\} = n^{\alpha + o(1)}$$

where $\alpha = \underbrace{\frac{1}{2}(3-\sqrt{2})}_{0.7929\ldots} = \underbrace{\text{Hausdorff dimension of } CLE_6 \text{ w.r.t. } \sqrt{2}\text{-LQG metric.}}_{\text{Computed using the KPZ relation}}$

In particular, there should be no infinite noodle but many macroscopic loops.

## 3.4 - Rigorous results on meandric systems

Let

$$d = d_{\sqrt{2}} := \text{Hausdorff dimension of the } \sqrt{2}\text{-LQG sphere}$$

[Intuitively this means that a Ball of radius $r$ has measure $\approx r^d$]

The number $d$ exists and is a.s. constant thanks to [Gwynne, Pfeffer, '22]. The exact value is NOT known (this is one of the most important open problems in the field) but [Gwynne, Pfeffer, '19] proves that

$$3.55 \approx 2(9+3\sqrt{5}-\sqrt{3})(4-\sqrt{15}) \leq d \leq \frac{2}{3}(3+\sqrt{6}) \approx 3.63.$$

Moreover, the fact that $(M_n, \Gamma_n) \longrightarrow (\sqrt{2}\text{-LQG}, SLE_8)$ in the Peano sphere sense, gives that

**Proposition:** W.h.p. $\operatorname{diameter}(M_n) = n^{1/d + o(1)}$ (w.r.t. the graph distance in $M_n$), that is, $\forall\, \xi \in (0,1)$

$$\mathbb{P}\left( n^{1/d - \xi} \leq \operatorname{diameter}(M_n) \leq n^{1/d+\xi} \right) \xrightarrow[n\to\infty]{} 1$$

The proposition is proved using standard techniques and follows from an important result of [Gwynne, Holden, Sun, '20] that allows to compare distances between planar maps and the limiting LQG-surface when one has convergence in the Peano-sphere sense.

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We are finally ready to state our main results:

**THEOREM 1** (B., Gwynne, Park '23)

W.h.p., $\exists\ \ell \in \Gamma_n$ such that $\operatorname{diameter}(\ell) \geq n^{1/d - o(1)}$

**Remarks:**

- The theorem above proves (modulo the $o(1)$ correction in the exponent) that loops in $\Gamma_n$ do not collapse to points in the limit, i.e. there are macroscopic loops.
- The $o(1)$ term is there because there are no available techniques to get up-to-constants bounds for graph-distances in $M_n$.
- $\operatorname{diameter}(\ell) \geq n^{1/d - o(1)} \implies \#\{\text{vertices in } \ell\} \geq n^{1/d - o(1)}$

- $\mathrm{diameter}(\ell) \geq n^{1/d - o(1)} \implies \#\{\text{vertices in } \ell\} \geq n^{1/d - o(1)}$

- $0.275 \leq 1/d \leq 0.282$ from the previous bounds on $d$.

- This is the first power-law bound (before Kargin proved $\log(n)$).

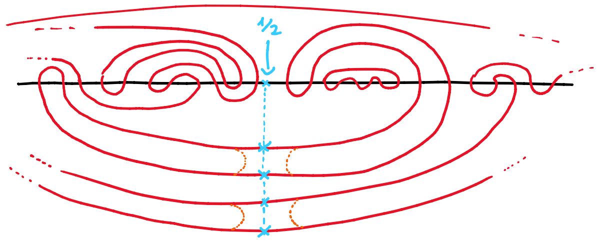
- $n^{1/d}$ is quite far from the previously conjecture $n^{\alpha}$, $\alpha \approx 0.79$:
  In order to get this improvement, one needs to show that the loop $\ell$ is much longer that a geodesic in $M_n$ (which requires a much finer understanding of the geometry of $\ell$ w.r.t. $M_n$).

- There are two recent works:
  - Duminil-Copin, Glazman, Peled, Spinka, 21
  - Crawford, Glazman, Harel, Peled, 20

  which prove existence of macroscopic loops for the critical $O(n)$-loop model on the hexagonal lattice in two different ranges for the parameter $n$.
  The results in these works and our results are quite similar in spirit, but the proofs are completely different (we are working on a random lattice).

**THEOREM 2** (B., Gwynne, Park, '23)

Consider the infinite meandric system $m_\infty$:



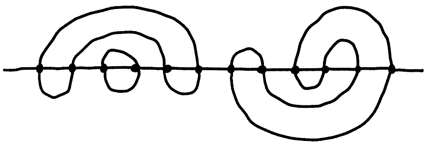

Cut all the arcs crossing the blue ray above (there are infinitely many arcs to cut) and rewire successive pairs of unmatched ends as shown in the picture (orange dotted lines).
Then, a.s., there are NO infinite loops.

**Remark:** The new m.s. above is called "Uniform Infinite Half Plane Meandric System" (UIHPMS) and it is conjectured to have exactly the same scaling limit of a uniform m.s. but with the topology of the half plane (instead of the sphere).

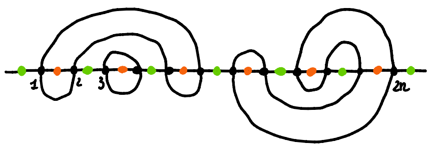
**Remark:** Morally, this is equivalent to say that there is no infinite cluster in percolation after removing a ray from the origin in the plane.

## 3.5 Meandric systems & percolation

Consider a meandric system:

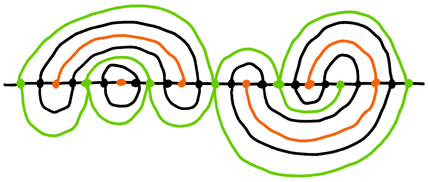


Color the half integers alternatively in green and orange:




Note that
green $= x - \frac{1}{2}$ with $x \in \mathbb{Z}$ odd
orange $= x - \frac{1}{2}$ with $x \in \mathbb{Z}$ even

Now, if possible, match each green (resp. orange) vertex with the closest green (resp. orange) vertex on the left and on the right that is matchable without crossing black arcs:



Note that each orange/green vertex has 4 potential half-arcs:

**Claim:**

① The configuration of green (resp. orange) arcs determines the black arcs and the orange (resp. green) arcs. (i.e. it determines the m.s.)

② If for each green vertex we independetly sample each of the four potential half-arcs uniformly and indep. at random, then we match half-arcs into arcs in the unique possible planar way, & finally we determine the black arcs configuration as in ①, then we get exactly a UIMS $M_\infty$.

**Proof:** Exercise.

In particular, we have the following percolation point-of-view:

green as open edges / orange as closed edges / black as boundaries of clusters.

## 3.6 - Proof of THEOREM 1 (Existence of macroscopic loops)

The proof of THEOREM 1 is divided into two main steps:

- STEP 1: Show that $\exists$ a loop in $\Gamma_n$ which is big w.r.t. the metric induced by $\Gamma_n$.
- STEP 2: SLE/LQG argument to lower-bound distances in $\Gamma_n$.

↳ Here we use some fundamental results:

- * Existence of the LQG-metric (Ding-Dubedat-Dunlap-Falconet + Gwynne, Miller)
- * Mating of trees (Sheffield-Miller-Duplantier)

I will only explain STEP 1. STEP 2 is quite standard (but you need to know quite a lot about SLEs and LQG).

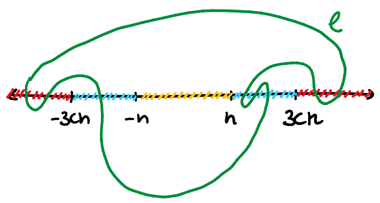
The main result in STEP 1 is the following theorem about the infinite m.s. $m_\infty$:

**Theorem:** For $n$ large enough, the following is true:

For every constant $C>1$, at least one of the following events holds with prob. $\geq \frac{1}{10}$:

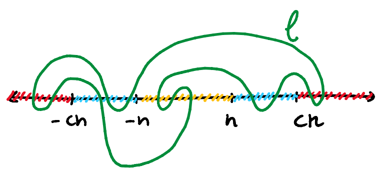
(A) $\exists$ loop $\ell$ such that:

- $[-n,n]$ is disconnected from $\infty$ by $\ell$
- $\ell$ hits $[-3Cn,-n) \cup (n,3Cn]$




(B) $\exists$ loop $\ell$ (or an infinite path $\ell$) such that:

- $\ell$ hits $[-n,n]$
- $\ell$ hits $\mathbb{Z} \setminus [-Cn,Cn]$



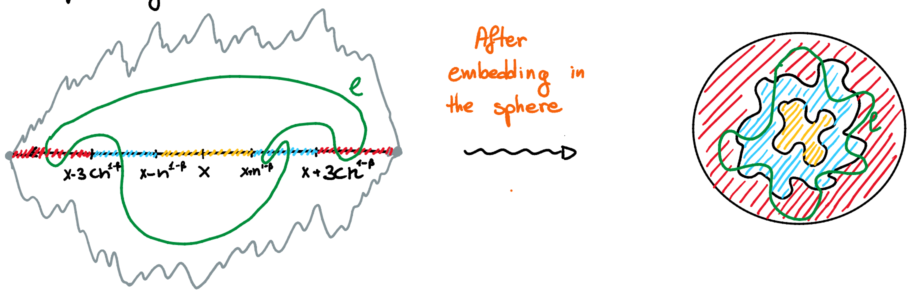

INTUITIVELY: Why is this enough to conclude that there are MACROSCOPIC LOOPS?

SKETCH: Using the independence between the event above and the same event translated to the right, we can ensure that the event above (with $n^{1-\beta}$ in place of $n$) happens somewhere in $[0,2n]$ with probability

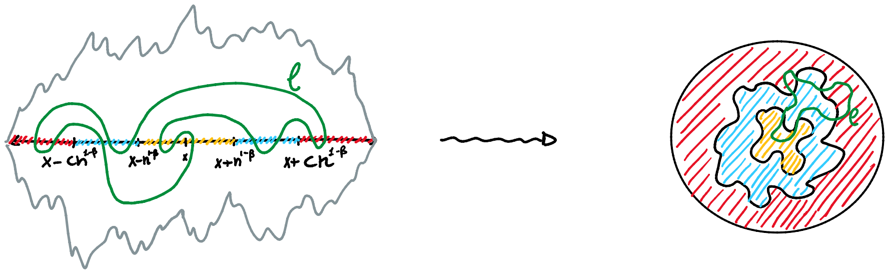
$$1-a_0 e^{-a_1 n^{\beta}}$$

for some $a_0, a_1, \beta > 0$. Then we look at this place where the event holds and we conclude as follows: We condition on the event $E_n$ that the two walks take an excursion between $0$ and $2n$ (this event has probability $\approx a_3 \cdot n^{-\rho}$, for some $\rho > 0$). Conditioning on $E_n$, $m_\infty$ between $0$ and $2n$ is a uniform meandric system of size $n$.

Hence with conditional probability at least $1 - a_0 \cdot a_3^{-1} \cdot n^{\rho} \cdot e^{-a_1 \cdot n^{\beta}}$ given $E_n$ we get that one of the following two events hold:




QUESTION: Why do we also need to ensure that $\ell$ intersects the blue region?
ANSWER: Because we do not want that the loop becomes tiny in the other side of the sphere!

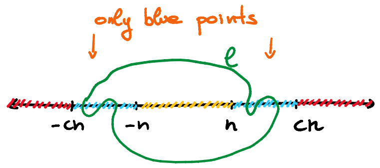



If $C$ is big enough, one can show that the 3 regions (red, blue, yellow) are all macroscopic in the limit and so conclude that $\ell$ is also macroscopic (in the first case $\ell$ must surround the yellow region and cannot become tiny in the other side of the sphere; while in the second case $\ell$ must traverse the blue region).

## Proof of the theorem:

Note that if $\exists$ an infinite loop in $m_\infty$ then the theorem is trivially true. So for the rest of the proof we are going to assume that a.s. $m_\infty$ has no infinite path.

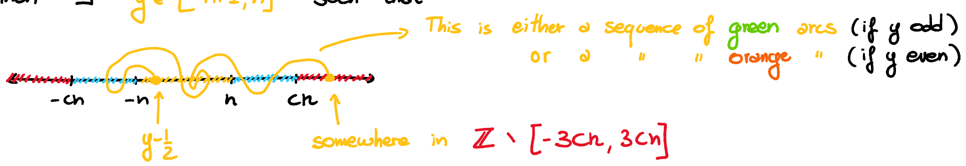
We consider the following event:

$E_n := \{ \exists$ loop $\ell$ such that  $\}$


If $\mathbb{P}(E_n) \geq \frac{1}{10}$ then we are done since (A) holds. Hence we assume that

$$\mathbb{P}(E_n^c) > 1 - \frac{1}{10} \quad \& \quad E_n^c \text{ holds}$$

If $E_n^c$ holds then $\exists$ $y \in [-n+1, n]$ such that



**Proposition:** With probability $\geq \frac{1}{10}$, $\exists$ loop such that:

① $\ell$ disconnects $y-\frac{1}{2}$ from $\infty$ for some $y \in [-n+1, n]$

② $\ell$ hits a point in $\mathbb{Z} \setminus [-Cn, Cn]$

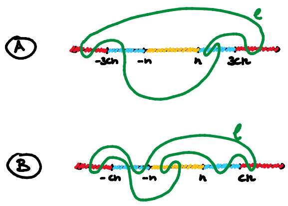
③ $\ell$ hits a point of $[y, 3Cn]$

**Why is this enough to conclude?**

- If $\ell$ disconnects $[-n,n]$ from $\infty$ then ③ $\Longrightarrow$ Ⓐ
- If not, then ① + ② $\Longrightarrow$ Ⓑ

Indeed these two conditions together imposes that $\ell$ must hit a point in $[-n,n]$

REMINDER:

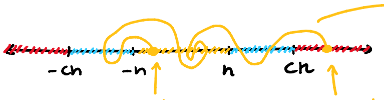



**Proof of the Proposition:**

Recall that we are assuming that

$$\mathbb{P}(E_n^c) > 1 - \frac{1}{10} \quad \& \quad E_n^c \text{ holds}$$

and if $E_n^c$ holds then $\exists$ $y \in [-n+1, n]$ such that




This is either a sequence of green arcs (if $y$ odd) or a " " orange " (if $y$ even)

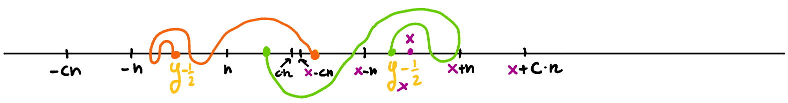
$y-\frac{1}{2}$ somewhere in $\mathbb{Z} \setminus [-3Cn, 3Cn]$

Now w.l.o.g. we can assume that $\mathbb{P}(y \text{ even} \mid E_n^c) \geq \frac{1}{2}$.

As a consequence, we can take $x \in [2Cn+1, 2Cn+3]$ odd such that:

$$\mathbb{P}\Big( \underbrace{E_n^c \cap \{y \text{ even}\}} \cap \underbrace{E_n^c(x) \cap \{y_x \text{ odd}\}} \Big) \geq \frac{\mathbb{P}(E_n^c)^2}{4}$$

↳ independent ↲



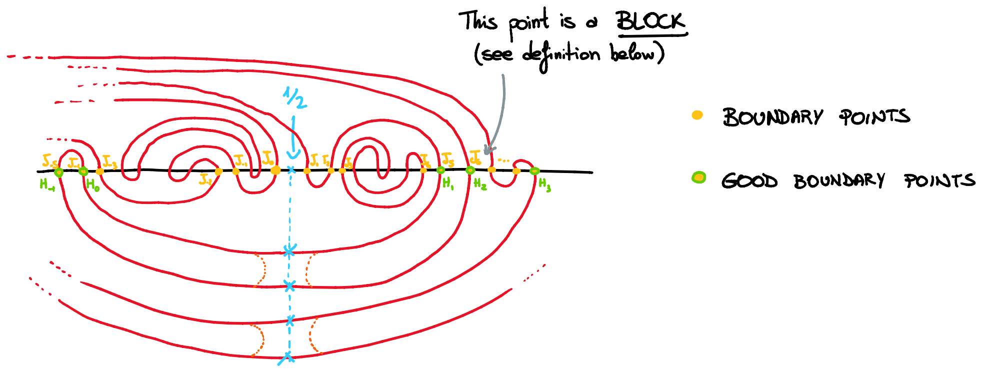

$y-\frac{1}{2}$ is orange and $y_x - \frac{1}{2}$ is green $\Longrightarrow$ $\exists$ loop $\ell$ separating $y$ and $y_x$.

- Then such loop $\ell$ must hit a vertex in $[y, y_x] \subseteq [y, 3Cn]$ $\Longrightarrow$ ③ holds.
- Moreover, with probability at least $\frac{\mathbb{P}(E_n^c)^2}{8} \geq \frac{1}{10}$, $\ell$ disconnects $y-\frac{1}{2}$ from infinity $\Longrightarrow$ ① and ② hold. □

↳ because $E_n^c$ holds.

## 3.7 Proof of THEOREM 2

Recall the UIHPMS:




**Def**: A point $x \in \mathbb{Z}$ is called a BOUNDARY POINT if it can be connected to the blue ray $\frac{1}{2} \times (-\infty, 0]$ by some continuous path without crossing any arc or the real line.
A point $x \in \mathbb{Z}$ is called a GOOD BOUNDARY POINT if it is a boundary point and is also an end-point of the rewired arcs.

We denote the boundary points by $J_k$, $k \in \mathbb{Z}$ so that $J_0 = 0$ (see picture above).
" " " good bound. points " $H_m$, $m \in \mathbb{Z}$ so that $0 \in [H_0, H_1 - 1]$ (see picture above).

**Observation 1**: Note that if $x = J_{2k} \in 2\mathbb{Z}$ is a boundary point then $x+1 = J_{2k+1} \in 2\mathbb{Z}+1$ is also a boundary point.
As a consequence, the law of the UIHPMS is invariant under $J_{2k}$-translations, for all $k \in \mathbb{Z}$.

**Observation 2**: The fact that a boundary point is a good boundary point is independent of the UIHPMS (see our paper for a careful explanation of this fact.)

**Definition**: For each good boundary point $H_m$, let $P_m$ be the unique directed path of arcs in the UIHPMS starting from $H_m$, following the arc incident to $H_m$ in the upper half-plane and ending at the first good boundary point other than $H_m$ (if it exists), otherwise let $P_m$ be the whole semi infinite path started from $H_m$.
We call $P_m$ the BOUNDARY PATH started from $H_m$.

**Lemma**: Almost surely, the boundary path $P_m$ is finite, $\forall m \in \mathbb{Z}$.

To prove the lemma we need one more definition:

**Def:** We call $x \in 2\mathbb{Z}_{>0}$ a UPPER BLOCK if

$x$ is linked with an upper-arc to a point in $(-\infty, 0]$.

If, in addition, $x = J_{2k}$ for some $k \in \mathbb{Z}$, we call $x$ a BLOCK.

See the point $J_6$ in the figure above for an example.

**Fact:** Fix $\varepsilon > 0$. Almost surely, there are infinitely many $k > 0$ such that $J_{2k}$ is a block and there are at most $\varepsilon \cdot k$ upper blocks in $(0, J_{2k}]$.

**Brief explanation:** One can show using some random walk estimates that $\mathbb{P}(A_n) \geq \frac{c}{n}$, where

$A_n = \{ 2n = J_{2k}$ for some $k \geq \frac{\sqrt{n}}{2}$ & there are at most $\varepsilon \cdot \sqrt{n}$ upper blocks in $(0, J_{2k}] \}$

Then one concludes that $\mathbb{P}(A_n \text{ i.o.}) > 0$ by Kochen-Stone theorem. Finally $\mathbb{P}(A_n \text{ i.o.}) = 1$ by 0-1-laws.

**Proof of the lemma:**

We first look at boundary paths $P_m$ started at good boundary points of the form $J_{2k}$, for some $k \in \mathbb{Z}$.
(Recall that the UIHPMS is invariant under $J_{2k}$-translations)

We set

$$E_k := \{ J_{2k} = H_m \text{ for some } m \in \mathbb{Z} \ \& \ P_m \text{ is semi-infinite} \}$$

Note that $\mathbb{P}(E_k) = p$ for all $k \in \mathbb{Z}$ (translation invariance + $J_{2k}$ is good is indep. of the UIHPMS)

Assume for a contradiction that $p > 0$. By the Birkhoff ergodic theorem:

$$\lim_{k \to \infty} \frac{1}{k} \# \{ k' \in [1,k] \cap \mathbb{Z} : E_{k'} \text{ occurs} \} = p \quad \text{almost surely.}$$

By this & the Fact above, we have that almost surely, there exist arbitrarily large $k \in \mathbb{Z}_{>0}$ s.t.

from Fact with $\varepsilon = p/4$
(i) $J_{2k}$ is a block;
(ii) there are at most $p/4 \cdot k$ upper blocks in $(0, J_{2k}) \cap \mathbb{Z}$;
(iii) $\# \{ k' \in [1,k] \cap \mathbb{Z} : E_{k'} \text{ occurs} \} > p/2 \cdot k$.

Fix $\kappa \in \mathbb{Z}_{>0}$ with the above 3 properties.

For $\kappa' \in (0,\kappa) \cap \mathbb{Z}$ such that $E_{\kappa'}$ occours, let $m(\kappa') \in \mathbb{Z}$ such that $H_{m(\kappa')} = J_{2\kappa'}$.

$P_{m(\kappa')}$ is semi-infinite, so it must exit $(0, J_{2\kappa})$ and it can do it only through an upper-arc connecting $(0, J_{2\kappa})$ to $(-\infty, 0]$ because $J_{2\kappa}$ is a block & there are no lower-arcs crossing $\frac{1}{2} \times (-\infty, 0]$.

But these upper arcs are then upper-blocks by definition. Hence we have at least $\frac{p}{2} \cdot \kappa$ upper blocks by (iii). A contradiction with (ii). Hence $p = 0$.

Finally, one repeats the same argument for points of the form $J_{2\kappa+1}$. $\square$

---

The last proved lemma gives a natural matching of the good boundary points $\{H_m\}_{m\in\mathbb{Z}}$

$H_m$ is matched with $H_{m'}$ iff $P_m$ ends at $H_{m'}$.

One can show that this matching is a <u>strongly ergodic random non-crossing perfect matching</u>.

The GOOD BOUNDARY POINTS are needed here!

The matching is translation invariant <u>for any translation</u>!
Any event wich is invariant under <u>even shifts</u> has probability zero or one.

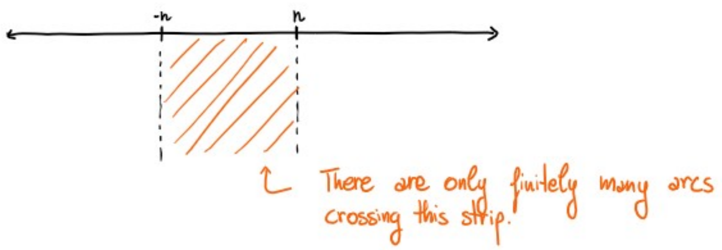
This implies (after some work) the following result:

**Lemma:** Almost surely, there exists infinitely many $m \in \mathbb{Z}_{>0}$ such that the boundary path $P_m$ ends at $H_{m'}$ for some $m' \in \mathbb{Z}_{\leq 0}$.

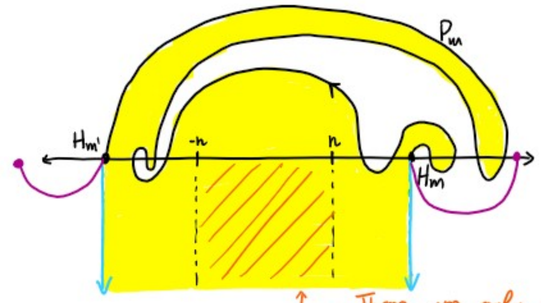
We can now complete the proof of the THEOREM 2:

Fix $n \in \mathbb{N}$. We will show that a.s. there is no bi-infinite path of arcs in the UIHPMS which visits $[-n,n]$.

Note that




By our lemma above, we can then find $m$ large enough such that $P_m$ ends at $H_{m'}$ for some $m' \in \mathbb{Z}_{<0}$ and $P_m$ never visits $[-n,n]$:




Note that there are no arcs crossing the two blue rays below $H_m$ and $H_{m'}$ by definition of boundary points & by definition of the UIHPMS.

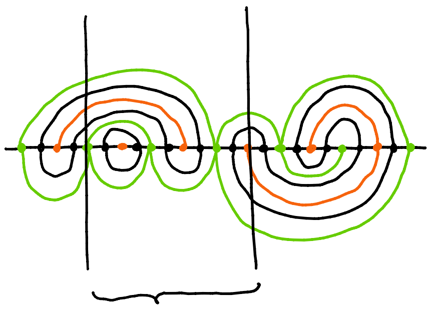
There are only finitely many arcs crossing this strip.

Now, note that a bi-infinite path needs to enter and exit the yellow region in order to visit $[-n,n]$ and this is possible only throug the lower arcs connected to $H_m$ and $H_{m'}$ (see the two purple arcs). But then, by construction of $P_m$ we would not touch $[-n,n]$. Hence there is no bi-infinite path crossing $[-n,n]$. □

## 3.8 Open questions

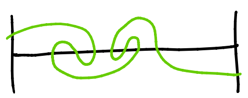
Here are some further open problems on meandric systems that I like:

Given a m.s. one can consider the following boxes starting at green and ending at orange:



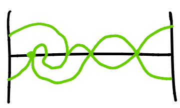
A box of size 5 (= number of black dots inside the box)

Note that each box has a top-left / top-right / bottom-left / bottom-right side.
In our paper we showed that the probability that $\exists$ a top-left to bottom-right green crossing is:

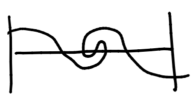
$\mathbb{P}\Big($  $\Big) = \frac{1}{2}$

**Question 1**: Can we compute the same crossing event for loops instead of green-paths?
**Question 2**: Is there a good notion of FKG for this model?

**Remark**: Crossing events seems to be anti-correlated, that is:

$\mathbb{P}\Big($  $\Big) \leq \frac{1}{4}$

I guess this would immediately gives the following lower bound for Question 1:

$\mathbb{P}\Big($  $\Big) \geq \frac{1}{4}$

## 4 The meandric permuton: A new scaling limit for uniform meanders

### 4.1 - Permutons

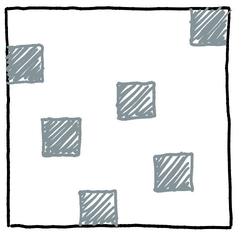
**Definition:** A PERMUTON $\mu$ is a probability measure on the unit square $[0,1]^2$ with uniform marginals, i.e. $\mu([a,b]\times[0,1]) = \mu([0,1]\times[a,b]) = b-a \quad \forall\ 0\le a\le b\le 1$.

There is a natural way to associate a permutation with a permuton.

**Example:** If

$$\sigma = 5\ 3\ 1\ 4\ 2\ 6 \qquad \text{(one-line notation)}$$

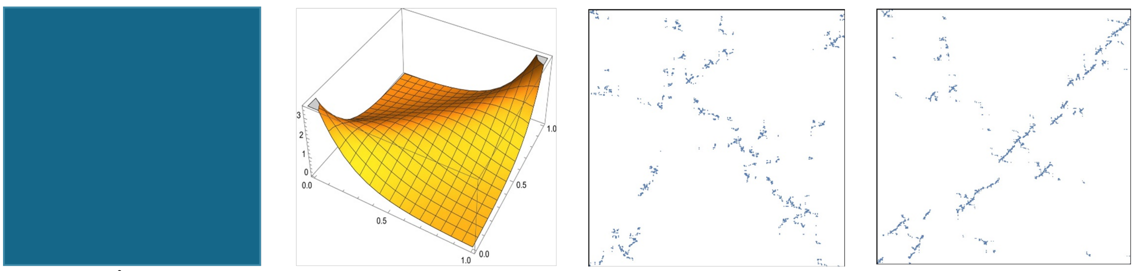
then

$\mu_\sigma =$  (Lebesgue measure on the gray support of total mass 1)

We then have a natural notion of convergence for permutations:

$$\sigma_n \xrightarrow{n\to\infty} \mu \iff \mu_{\sigma_n} \xrightarrow{n\to\infty} \mu \quad \text{w.r.t. weak-topology.}$$

**Some interesting permutons:**



Uniform — Mallows — Pattern-avoiding

**A BIT OF HISTORY:**

- Introduced by Hoppen, Kohayakawa, Moreira, Rath, Sampaio in 2011 to study permutation sequences and parameter testing (as a counterpart of graphons for graph).

- Popularized in the probabilistic community by:

  * Kenyon, Kral, Radin, Winkler ⇝ Permutations with fixed pattern densities.
  * Bassino, Bouvel, Féray, Gerin, Maazoun, Pierrot ⇝ Limits of pattern-avoiding permutations

- There is now a quite large literature studying limits of non-uniform random permutations (see, for instance, my PHD thesis for an overview).

## 4.2 - An instructive example: The case of Baxter permutations

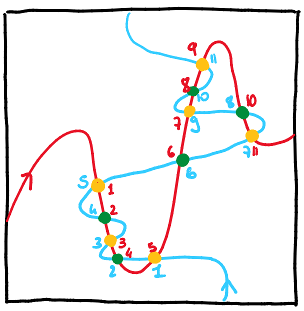
**Def:** A MONOTONE MEANDER of size $n$ is a pair of simple curves $\ell_1$ and $\ell_2$ in $[0,1]^2$ which cross each other exactly $2n-1$ times and such that:

- $\ell_1$ starts on the left-hand side of $[0,1]^2$, ends on the right-hand side of $[0,1]^2$ and never moves in the left direction. (Equivalently, it is the graph of a cont. function from $[0,1]$ to $[0,1]$.)

- $\ell_2$ starts on the bottom side of $[0,1]^2$, ends on the top side of $[0,1]^2$ and never moves in the bottom direction. (Equivalently, it is the graph of a cont. function from $[0,1]$ to $[0,1]$ rotated by 90 degrees).

We identify two monoton meanders using the usual identification for meanders.

**Example:** There is a natural way to encode a monotone meander with a permutation:



$\longrightarrow \quad \sigma = 5\ 4\ 3\ 2\ 1\ 6\ 9\ 10\ 11\ 8\ 7$

$\varphi$

Reduce the yellow permutation to the unique permutation on $\{1,\dots,n\}$ preserving the relative order between numbers.

$\pi = 3\ 2\ 1\ 5\ 6\ 4$

"Def": The set $\varphi$ (Monotone meanders) is called the set of Baxter permutations.

Proposition: The map $\varphi$ is a bijection between motone meanders of size $n$ and Baxter permutations of size $n$.

Baxter permutations enjoys several nice combinatorial properties & equivalent definitions:

Def: A BAXTER PERMUTATION is a permutation avoiding the patterns 2-41-3 and 3-14-2

i.e. $\sigma$ is Baxter $\iff \nexists\ i_1<i_2<i_3$ s.t. $\sigma(i_2+1)<\sigma(i_1)<\sigma(i_3)<\sigma(i_2)$ or $\sigma(i_2)<\sigma(i_3)<\sigma(i_1)<\sigma(i_2+1)$

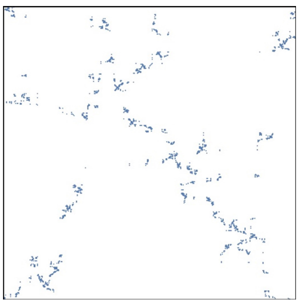
Theorem: (Borga, Maazoun, 2021)

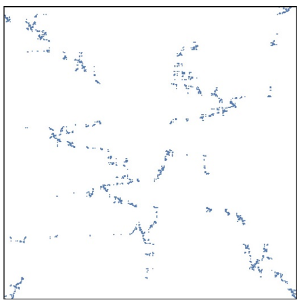
If $\sigma_n$ is a uniform Baxter permutation of size $n$, then

$$\mu_{\sigma_n} \xrightarrow[n\to\infty]{d} \mu_B =: \text{Baxter permuton}$$

Remark: The Baxter permuton $\mu_B$ is a random permuton, i.e. a random probability measure of the unit square $[0,1]^2$.

Simulations:





Baxter permutations

IMPORTANT COMMENSTS:

- These permutations are in bijection with other interesting combinatorial objects, such as trees, walk in cones, bipolar orientations, tessellations, etc.
- The proof of the theorem above involves the study of certain coalescent-walk processes.

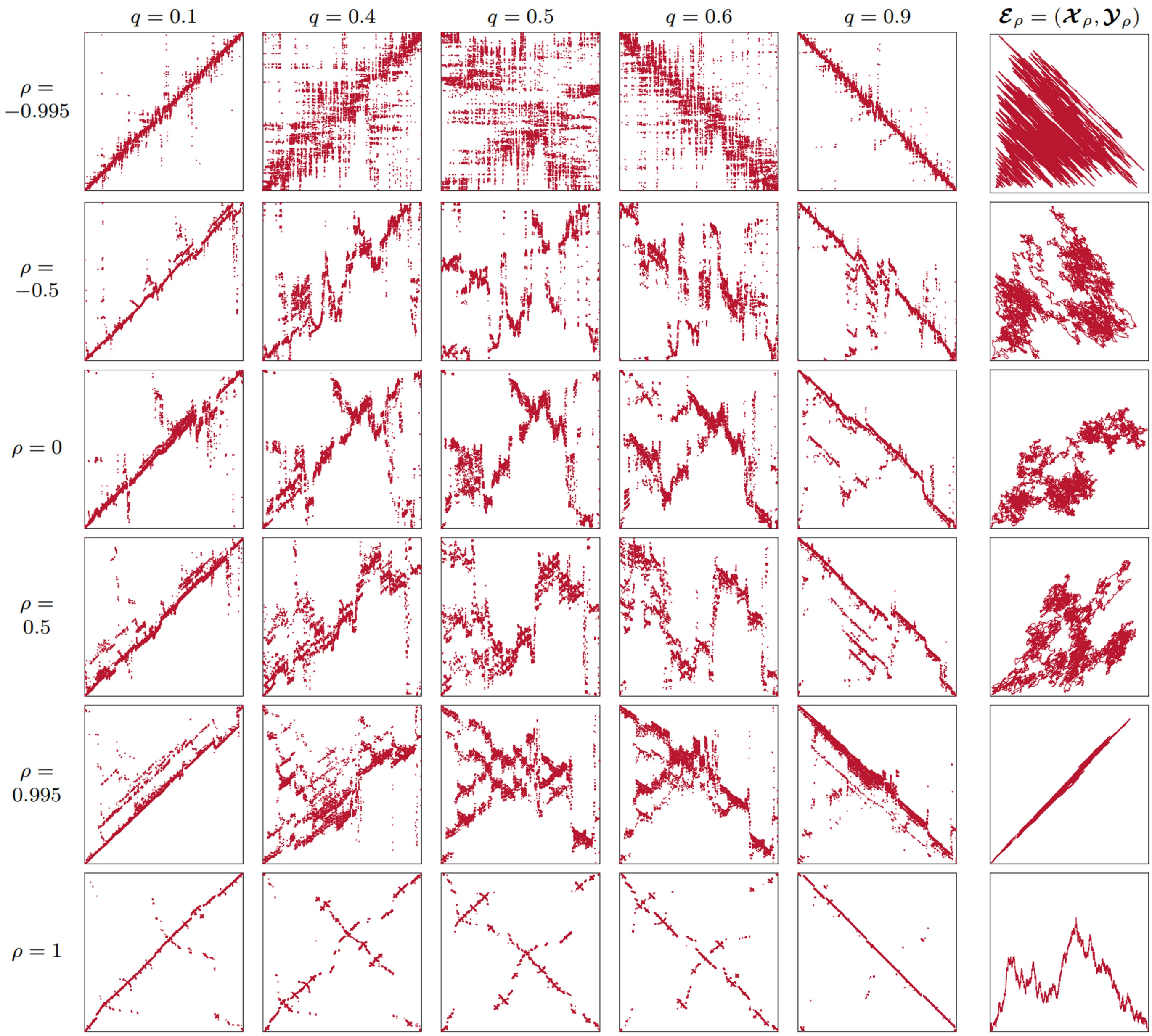
- The Baxter permuton can be constructed from a two-dim. correlated Brownian exc.
  \+ a system of SDEs.

- I recently introduced a two-parameter family $(\mu_{\rho,q})_{\rho\in[-1,1],\, q\in[0,1]}$ of random permutons, called skew Brownian permutons which are universal objects (i.e. scaling limits of many models of random permutations) and such that

$$\mu_{-\frac{1}{2},\frac{1}{2}} = \mu_B \quad \& \quad \mu_{1,q} = \text{Brownian separable permuton } (q) \quad \forall q\in[0,1]$$

↳ Another important family of random permutons.

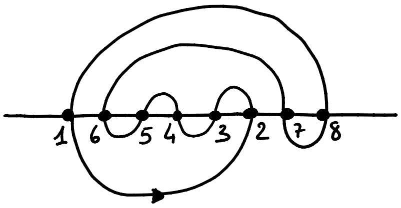
Simulations:



## 4.3 - The meandric permuton (PART 1)

We can encode a meander with its "meandric permutation" in a similar way:

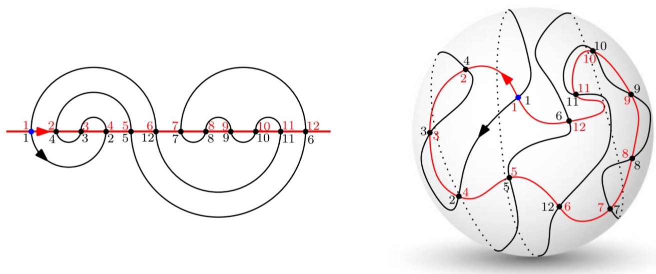
**Example:**

meander = 


$\xrightarrow{\varphi}$ meandric permutation $= 1\,6\,5\,4\,3\,2\,7\,8$

**Def:** A MEANDRIC PERMUTATION of size $n \in \mathbb{N}$ is a permutation that can be obtained from the map $\varphi$.

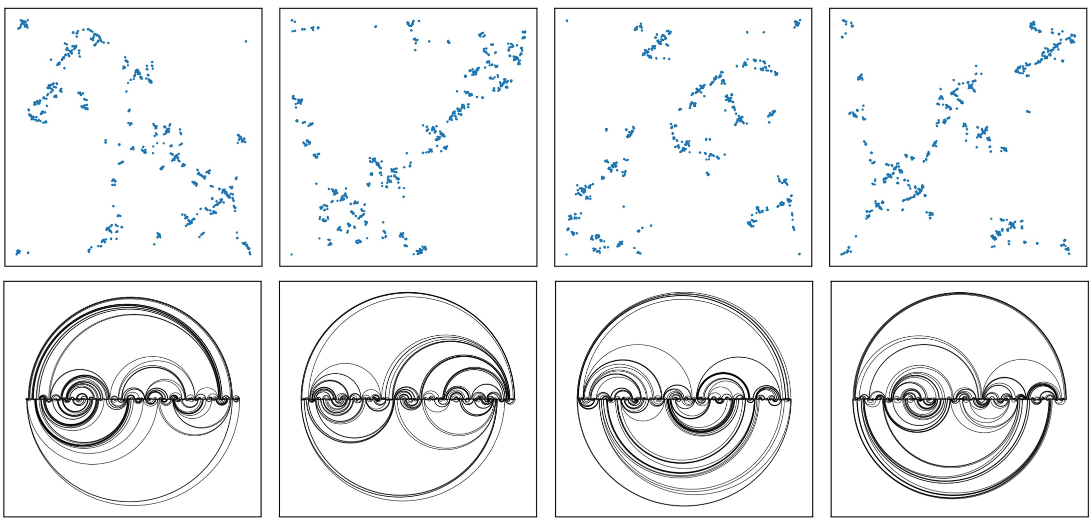
Sometime it is convenient to think about meanders on the sphere:



**"Definition:"** The MEANDRIC PERMUTON is the conjectural scaling limit of uniform meandric permutations.

Later we will propose an explicit construction of the meandric permuton.

**Simulations:**



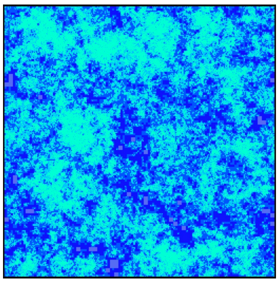
## 4.4 - Some important objects in Random geometry: SLEs & LQG

We only give informal definitions here:

**Definition:** For $\gamma \in (0,2)$, a $\gamma$-LQG area measure $\mu$ is a random probability measure on $\mathbb{C}$ obtained as the limit of regularized versions of

$$e^{\gamma \cdot h} d^2z, \quad \text{where} \quad \begin{cases} h \text{ is a whole-plane GFF} \\ d^2z \text{ is Lebesgue-measure on } \mathbb{C}. \end{cases}$$

[FOR EXPERTS: $\mu$ is the area-measure corresponding to a singly-marked unit-area $\gamma$-quantum sphere.]



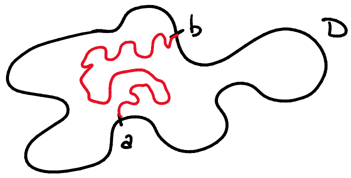
IMPORTANT FACTS about $\mu$:

- $\mu$ is random;
- $\mu(\mathbb{C}) = 1$ a.s.
- $\mu$ is a.s. non-atomic;
- $\mu$ a.s. assigns positive mass to open subsets.

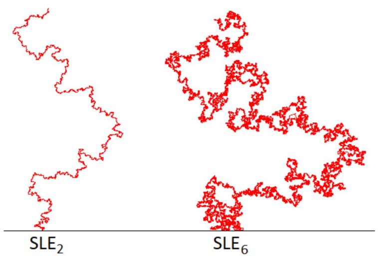
**Definition:** Let $D$ be a simply connected, open, complex domain not equal to $\mathbb{C}$. Fix $a, b \in \partial D$. A CHORDAL SLE from $a$ to $b$ in $D$ is a one-parameter family (indexed by $\kappa \geq 0$) of random non-crossing curves (viewed modulo time parametrization) in $D$ from $a$ to $b$ which satisfies:

× conformal invariance;

× a certain domain Markov property.




IMPORTANT FACTS about CHORDAL SLEs:

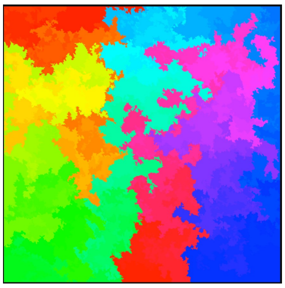



- These curves are non-crossing
- For $0 \leq \kappa \leq 4$ the curve is a.s. simple
- For $4 < \kappa < 8$ the curve a.s. touches itself & every point is contained in a loop
- For $\kappa \geq 8$ the curve is a.s. space-filling.

**Definition:** The whole-plane space-filling $SLE_{\kappa'}$ (from $\infty$ to $\infty$) is a random space-filling curve $\eta$ in $\mathbb{C}$ which:

- starts and ends at $\infty$;
- when $\kappa' > 8$, $\eta$ is just a two-sided variant of chordal $SLE_{\kappa'}$.
- when $\kappa' \in (4,8)$, $\eta$ can be obtained from a two-sided variant of $SLE_{\kappa'}$ by iteratively "filling-in" the loops which it disconects from its target point.

IMPORTANT FACTS about whole-plane SLEs:



- These curves are non-crossing but space-filling for all $\kappa' > 4$.
- These curves are invariant under scaling, translation, rotation & time-reversal.
- Given a $\gamma$-LQG $\mu$ and an independent whole-plane $SLE_{\kappa'}$ we always parametrize $\eta$ with $\mu$, i.e.

$$\mu(\eta[0,t]) = t, \quad \forall t \in [0,1].$$

## 4.5 - Constructing permutons from SLEs and LQG: The meandric permuton (PART 2)

The recipe:

Fix $\gamma \in (0,2)$ and $\kappa_1, \kappa_2 > 4$.

Let

INDEPENDENT

- $\mu$ be a $\gamma$-LQG area measure
- $(\eta_1, \eta_2)$ a pair of whole-plane space-filling SLEs of parameters $(\kappa_1, \kappa_2)$.

↳ The coupling between $\eta_1$ and $\eta_2$ is NOT specified for the moment.

Then

1) We parametrize $\eta_1$ and $\eta_2$ with $\mu$;

2) We consider the function $\psi : [0,1] \longrightarrow [0,1]$ such that $\eta_1(t) = \eta_2(\psi(t))$, for all $t \in [0,1]$.

↳ Informally, the "continuum permutation" obtained by comparing the order in which $\eta_1$ and $\eta_2$ hits the points of $\mathbb{C}$.

3) We define the permuton associated with $(\mu, \eta_1, \eta_2)$ by

$$\pi(A) := \underbrace{\mathrm{Leb}}_{\text{One-dimensional Leb. measure}} \{ t \in [0,1] : (t, \psi(t)) \in A \} \quad (*)$$

↳ Informally, $\pi$ is the "diagram" of the permutation $\psi$.

One can replace (*) with the following equivalent definition indep. of $\psi$:

**Lemma:** $\pi\left([a,b]\times[c,d]\right) = \mu\left(\eta_1([a,b]) \cap \eta_2([c,d])\right)$, for all rectangles $[a,b]\times[c,d] \subseteq [0,1]^2$.

**Proof:** By (*)

$$\pi\left([a,b]\times[c,d]\right) = \mathrm{Leb}\left\{ t\in[a,b] \mid \psi(t)\in[c,d]\right\}$$

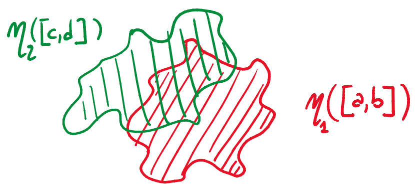
We need to be careful with multiple points of SLEs (but this is OK!)

Now, a.s., for almost every $t\in[0,1]$, we have

$$\psi(t)\in[c,d] \iff \eta_2(\psi(t)) \in \eta_2([c,d]).$$

Since $\eta_2(\psi(t)) = \eta_1(t)$ by definition, then

$$\pi\left([a,b]\times[c,d]\right) = \mathrm{Leb}\left\{ t\in[a,b] \mid \eta_1(t)\in \eta_2([c,d])\right\}$$




$$= \mu\left(\eta_1([a,b]) \cap \eta_2([c,d])\right)$$

$\hookrightarrow$ by our choice of time parametrization. $\square$

## Two important special cases:

- In Borga '22, it was shown that for each choice of $(\rho,q)\in(-1,1)\times(0,1)$ for the skew Brownian permuton $\mu_{\rho,q}$, there exists $\gamma\in(0,2)$ and a coupling of two whole-plane $SLE_{16/\gamma^2}$ curves such that the permuton constructed above coincides with $\mu_{\rho,q}$.

ONLY FOR EXPERTS: (Given $\gamma\in(0,2)$, the parameter $q\in(0,1)$ determines an angle $\theta\in(0,\pi)$ so that the two SLEs are coupled with the "Imaginary geometry coupling" of angle $\theta$.)

In a non-explicit way!

$\hookrightarrow$ Miller/Sheffield

- Togheter with Gwynne and Sun (2022), we conjectured that the permuton limit of meandric permutations is the permuton constructed above when

See our paper for further explanations.

$$\gamma = \sqrt{\tfrac{1}{3}\left(17-\sqrt{145}\right)}, \quad K_1 = K_2 = 8, \quad \eta_1 \text{ and } \eta_2 \text{ are independent}$$

and so we called it the MEANDRIC PERMUTON.

[Moreover, the map determined by a meander should converge to $\gamma$-LQG + 2 ⫫ $SLE_8$ and the "letters" should be "determined" by the MEANDRIC PERMUTON (work-in-progress with E. Gwynne)]

## SKEW BROWNIAN PERMUTONS

$SBP(\gamma, \theta)$

- $\eta_1$ and $\eta_2$ are (strongly) coupled $(\theta)$

  one determines the other one.

- $\gamma \in (0,2)$ & $\kappa_1 = \kappa_2 = 16/\gamma^2$

  (MATCHED MODELS $\kappa = 16/\gamma^2$)

- UNIVERSAL FAMILY of PERMUTONS

  (limits of family of permutations)

- 2D-Brownian motion construction

  (Scaling limits are doable (~~NOT~~ easy))

## PERMUTONS from INDEPENDENT SLEs

$IP(\gamma, \kappa_1, \kappa_2)$

- $\eta_1$ and $\eta_2$ are independent

- $\kappa_1, \kappa_2, \gamma \in (4,\infty)^2 \times (0,2)$

- Meandric permuton $= IP\left(8, 8, \sqrt{\tfrac{1}{3}(17-\sqrt{145})}\right)$

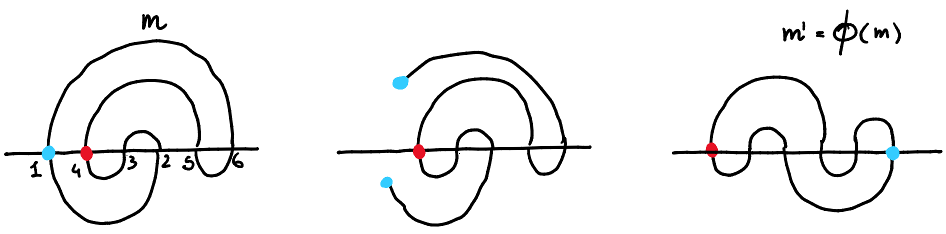
  (MISSMATCHED MODEL $\kappa \neq 16/\gamma^2$)

- No 2D-Brownian motion construction

  (Scaling limits are very difficult)

## 4.6 – Rerooting invariance for the meandric permuton

We consider the following re-rooting operation on meanders:




1 4 3 2 5 6

Open the meander on the left

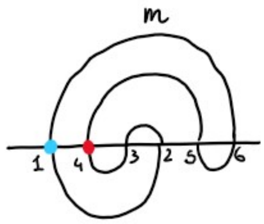
Close the meander on the right

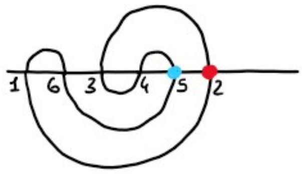
We get a new meander $m'$

**Proposition:** If $m$ is a uniform meander then $\phi(m)$ is a uniform meander.

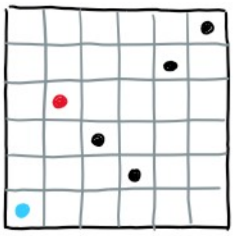
**Q:** How this property translates in terms of permutations?

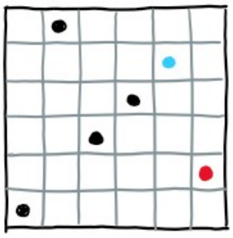
**Example:** We look at the permutations corresponding to m and $\phi^2(m)$:

$m =$ 

$\phi^2(m) =$ 

1 4 3 2 5 6

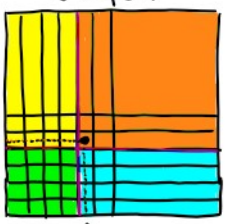
$\varphi(m) =$ 

$\varphi(\phi^2(m)) =$ 

To go from $\varphi(m)$ to $\varphi(\phi^2(m))$ we need to do the following operation:

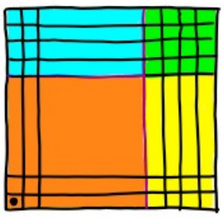
**Proposition:** Let m be a meander and $k \in \mathbb{Z}_{>0}$.
The diagram of the permutation $\varphi(\phi^{2k}(m))$ is obtained from the diagram of $\varphi(m)$ as follows: divide the diagram of $\sigma = \varphi(m)$ into these four rectangle

$\sigma = \varphi(m)$

$\sigma(2k+1) \rightarrow$ 

$\uparrow$
$2k+1$

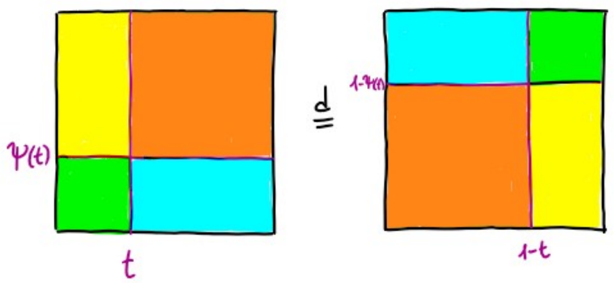
then $\sigma' = \varphi(\phi^{2k}(m))$ has the following diagram:

$\sigma' =$ 

**Exercise:** Prove the above proposition.

We can prove the same property for the MEANDRIC PERMUTON:

**Theorem** (B., Gwynne, Sun '23) Let $\pi$ be the meandric permuton. Fix $t \in [0,1]$. The following two permutons are equal in distribution:




where $\psi$ is the function used to define the meandric permuton.

- The proof of the theorem above builds on a certain rerooting invariance property of $SLE_8$.
- We call the above property "re-rooting invariance for permutons."
- We can prove that $IP(\gamma, 8, 8)$ is rerooting invariant for all $\gamma \in (0,2)$, but we will prove that $IP(\gamma, \kappa_1, \kappa_2)$ is NOT re-rooting invariant as soon as $\kappa_1 \neq 8$ or $\kappa_2 \neq 8$.

## 4.7 - Some goals for the future

- We would like to find a list of "natural properties" that uniquely characterize the law of the MEANDRIC PERMUTON.
- Re-rooting invariance is obviously one of these properties
- From the above discussions we now have a convincing explanation on why $\kappa_1 = \kappa_2 = 8$ for the meandric permuton.
- We are working on a "second property" that should be satisfied by $IP(\gamma, \kappa_1, \kappa_2)$ only for a unique value $\gamma^* = \gamma^*(\kappa_1, \kappa_2)$. In particular, we should be able to show that

$$\gamma^*(8,8) = \sqrt{\frac{1}{3}\left(17 - \sqrt{145}\right)}$$

which is exactly the value conjectured for the meandric permuton.

## 4.8 - Other results on PERMUTONS constructed from SLEs and LQG

In some joint works with Ewain Gwynne, Xin Sun, Nina Holden and Pu Yu, we started to investigate several properties of the permutons $\pi$ constructed in the previous sections. Here are some results:

(1) We explicitly computed $E[\mu_B]$, when $\mu_B$ is the Baxter permuton;
(2) We investigated the behaviours of patterns in the skew Brownian permuton $\mu_{\gamma,q}$;
(3) We showed that the Hausdorff dimension of the support of all permutons $\pi$ constructed in the previous section is always one;
(4) We studied the behaviour of the length of the longest increasing subsequence in permutations converging to the permutons constructed in the previous section.

All these results are obtained combining quite many results/techniques coming from the SLE/LQG literature. We now just focus on one specific result, i.e. (4).

## 4.9 - The length of the longest increasing subsequence

**Definition:** For a permutation $\sigma$, the length of the longest increasing subsequence $LIS(\sigma)$ is the maximal cardinality of a subset $L \subseteq [1, |\sigma|] \cap \mathbb{Z}$ s.t. the restriction of $\sigma$ to $L$ is monotone increasing.

**THEOREM:** (B., Gwynne, Sun, '22)

Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of random permutations s.t. $|\sigma_n| \longrightarrow \infty$ and whose associated permuton $\pi_n$ satisfies one of the following conditions:

(a) $\mu_{\sigma_n} \xrightarrow{d} IP(\kappa_1, \kappa_2, \gamma)$

(b) $\mu_{\sigma_n} \xrightarrow{d} SBP(\gamma, \theta)$

Then

$$\frac{LIS(\sigma_n)}{|\sigma_n|} \xrightarrow{P} 0.$$

## Comments:

(1) The theorem is saying that $LIS(\sigma_n) = o(|\sigma_n|)$.

(2) There exist families of pattern-avoiding permutations $\mathcal{C}$, s.t. if $\sigma_n$ is uniform in $\mathcal{C}$ then $LIS(\sigma_n) = \Omega(|\sigma_n|)$.

(3) The theorem holds for Baxter permutations.

(4) The law of the length of the longest increasing subsequence in Baxter permutation is equal to the law of the longest directed path in bipolar orientations. The latter can be interpreted as a model of <u>last passage percolation in planar maps</u>.
The scaling limit of this model should be a sort of <u>directed-LQG metric</u>, which should be the "quantum analogue" of the <u>directed landscape</u>.

<u>OPEN PROBLEM</u>: If $\sigma_n \longrightarrow SBP(\gamma, \theta)$, determine $\alpha(\gamma,\theta)$ s.t.

$$|\sigma_n| \sim C \cdot n^{\alpha(\gamma,\theta)}, \quad \text{as} \quad n \to \infty.$$

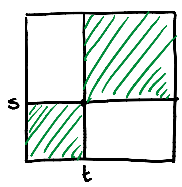
<u>Conjecture</u>:
- $\alpha(\gamma,\theta) = \alpha(\gamma)$, i.e. independent of $\theta$
- $\alpha(\gamma)$ is increasing in $\gamma$
- $\alpha(\gamma) \xrightarrow{\gamma \to 0} 1/2$ ⇝ This is the exponent for uniform permutations.
- $\alpha(\gamma) \xrightarrow{\gamma \to 1} 1$

## <u>Idea of the proof of the Theorem</u>:

We start with a definition:

<u>Definition</u>: We say that a set $A \subseteq [0,1]^2$ is <u>monotone</u> if

$$A \subseteq [0,t] \times [0,s] \cup [t,1] \times [s,1], \quad \forall (t,s) \in A.$$




Note that for instance the graph of a non-decreasing function (or a subset of it) is monotone.

**Proposition 1:** Let $\pi$ be a permuton and let $(\sigma_n)_n$ be a sequence of permutations of size $|\sigma_n| \to \infty$ whose associated permuton $\pi_{\sigma_n} \xrightarrow[n\to\infty]{} \pi$.
Assume that $\pi(A) = 0$ for every monotone set $A \subseteq [0,1]^2$. Then

$$\frac{LIS(\sigma_n)}{|\sigma_n|} \xrightarrow[n\to\infty]{} 0.$$

(The proof uses basic topological facts.)

**Proposition 2:** Let $X$ be a $\mu_h$-measurable set with the following property:
For $\mu_h$-a.e. pair of points $z, w \in X$, $\eta_1$ and $\eta_2$ hit $z$ and $w$ in the same order.
Then $\mu_h(X) = 0$.

(The proof is not too long but requires some deeper results for SLEs and LQG)

We now prove the theorem, assuming the two propositions hold.

By assumption, $\mu_{\sigma_n} \xrightarrow{d} \pi \; \big(= IP(\kappa_1, \kappa_2, \gamma) \text{ or } SBP(\gamma, \theta)\big)$
Recall the function $\psi : [0,1] \to [0,1]$ in the definion of $\pi$. In particular $\eta_1(t) = \eta_2(\psi(t))$, $\forall t \in [0,1]$.
For a monotone set $A \subseteq [0,1]^2$, let

$$T_A := \{ t \in [0,1] : (t, \psi(t)) \in A \}.$$

**Claim:** A.s. for every monotone set $A$, it holds that $\mu_h$-a.e. pair of points $z, w \in \eta_1(T_A)$ are hit in the same order by $\eta_1$ and $\eta_2$.

**Proof:** We can assume that $\eta_1$ and $\eta_2$ have no double points (since a.s. $\mu_h(\text{double points}) = 0$).
Fix $z, w \in \eta_1(T_A)$ and let $t_z, t_w \in [0,1]$ be the times s.t. $\eta_1(t_z) = z$, $\eta_1(t_w) = w$.
By def of $T_A$, $(t_z, \psi(t_z)) \in A \ni (t_w, \psi(t_w))$. W.l.o.g. we can assume that $t_z < t_w$, and then since $A$ is monotone, we get that $\psi(t_z) < \psi(t_w)$.
By def. of $\psi$, $\psi(t_z)$ and $\psi(t_w)$ are the times when $\eta_2$ hits $z$ and $w$.
Therefore $z$ and $w$ are hit in the same order by $\eta_1$ and $\eta_2$. □

Therefore

$$0 \overset{\text{Proposition 2}}{=} \mu_h(\eta_1(T_A)) \overset{\text{By parametrization of } \eta_1}{=} \text{Leb}(T_A)$$

Therefore, by definition of $\pi$, we get $\pi(A) = \text{Leb}(T_A) = 0$.

We proved that $\forall A \subseteq [0,1]^2$ monotone, then $\pi(A) = 0$. By Proposition 1 we can conclude the proof of the theorem. □