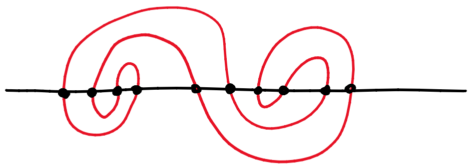
Meanders & Meandric systems

1. Introduction

Henry Poincaré, 1912:

"In how many ways a simple loop in the plane can cross a line at a specified number of points?"



self-avoiding simple curve

which we always fix to be $\{1,\dots,2n\}\subseteq\mathbb{Z}$

Definition: A meander of size n is a simple loop which crosses a strait line at $2n$ points.
A configuration is defined up to homeomorphisms of the plane which fix the line and map the upper-half plane into the upper-half plane.

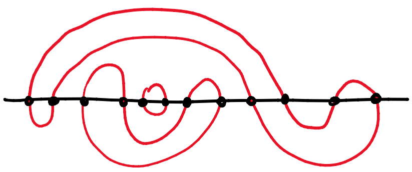
Let

$$M_n := \#\text{ of meanders of size } n$$

then the previous question can be rephrased as:

Can we determine M_n?

Definition: A meandric system of size n is a set of disjoint simple loops in the plane which cross a strait line at $2n$ points (+ same equiv. relation as above).



Remark: A meander is just a meandric system with one loop.

The number of meandric systems m_n of size n is <u>much</u> simpler to compute:

Note that the parts of the loops above (resp. below) the line form an arc diagram, i.e. a non-crossing perfect matching of the points $\{1, \dots, 2n\}$, and so

$$m_n = C_n^2 \quad \text{where} \quad C_n = \frac{1}{n+1}\binom{2n}{n} = n\text{-th Catalan number}.$$

In particular,

constant exponential polynomial

$$m_n \sim \left(\frac{4^n}{n^{3/2}\sqrt{\pi}}\right)^2 = \frac{1}{\pi}\, 16^n \cdot n^{-3}$$

What about M_n? Obviously,

$$C_n \leq M_n \leq C_n^2$$

Any top arc-diagram can be completed in at least one way into a meander. (Exercise)

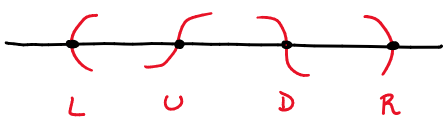
Beautiful survey paper on meanders / meandric systems

It is conjecture (in various papers; see "Meanders: A personal perspective" by Alexander Zvonkin) that

$$M_n \sim C \cdot A^n \cdot n^{-\alpha} \quad \text{for some } C, A, \alpha > 0.$$

(Same "structure" as meandric systems & many others comb. structures...)

Encoding every meander as a sequence in $\{L, U, D, R\}^{2n}$ by assigning:



and imposing that M_n must avoid "LR" one gets:

$$A \leq (2+\sqrt{3})^2 \approx 13.92.$$

And with a more sophisticated analysis, Albert & Paterson, 2005, obtained that

$$11.380 \leq A \leq 12.901$$

Numerical simulations (by Jensen & Guttmann) suggests that

$$A \approx 12.26287$$

To the best of my knowledge, there is no conjecture for the exact value of A or even for its nature.

Considering the generating function

$$M(t) = \sum_{n=0}^{\infty} M_n \cdot t^{2n}$$

and noting that $M_{a+b} \geq M_a \cdot M_b$, we get that

$$A := \lim_{n \to \infty} M_n^{1/n} \quad \text{exists}$$

and $A = \left(\frac{1}{\rho}\right)^2$ where ρ is the radius of convergence of $M(t)$.

To the best of my knowledge, this is all we rigourously know about the Poincaré's questions after more than 100 years....

But the story becomes even more interesting around 2000, when a remarkable physics paper by Di Francesco, Golinelli, and Guitter conjectured (again) that

$$M_n \sim C \cdot A^n \cdot n^{-\alpha}$$

but they additionally conjectured the exact value for α:

$$\alpha = \frac{29 + \sqrt{145}}{12} \approx 3.420132882$$

Numerical simulations (again by Jensen & Guttmann) confirms this prediction (Simulations for the exponent α match with the conjectured value up to three digits after the comma).

A natural question is:

"Why the conjecture is for α not for A?"

In some sense, α (→ sometime called the "CRITICAL EXPONENT") is more important than A:

- In physics, α controls the type of phase transition.
- In combinatorics, α controls the type of singularity of $M(t)$ at the radius of convergence.

Di Francesco, Golinelli, and Guitter arrived to the above conjecture combing two ingredients:

1. Describe the 'central charge c' of a limiting Conformal Field Theory for meanders $(c=-4)$
2. Use the KPZ-equation (Knizhnik-Polyakov-Zamolodchikov) to derive α from c.

Motivations: On top of being simple & beautiful comb. objects, meanders & meandric systems have been used in physics as models for polymer folding.

Moreover these objects are related to many areas of mathematics:

- COMBINATORICS
- PROBABILITY
- GEOMETRY OF MODULI SPACES OF SQUARED TILED SURFACES (Delecroix, Goujard, Zograf, Zorich, 2019)
- COMPLEX ANALYSIS
- ...

2. The plan of this mini-course

In this mini course we will mainly focus on describing the geometry of uniformly random meanders & meandric systems:

- In the 1st part we will focus on meandric systems:

    1- We will first present a new conjecture describing the limiting geometry for meandric systems.

    2- We will present several rigourous results which are consistent with the conjecture above.

- In the 2nd part we will focus on meanders:

    1- We will propose (a surprising) limiting object for meanders, called THE MEANDRIC PERMUTON

    2- We will prove some interesting properties of this new object.

If I have some time at the end, I also plan to explain a bit more the connections between the physics conjectures mentioned above and our conjectures. The curious reader can have a look at:

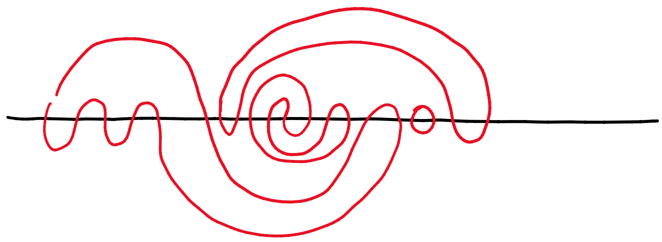
× "Permutons, meanders, & SLE-decorated Liouville quantum gravity". Borga, Gwynne, Sun.
  ↳ Section 6

× "On the geometry of uniform meandric systems". Borga, Gwynne, Park.
  ↳ Section 7

3. On the geometry of uniform meandric systems

3.1 Previous works on meandric systems

Recall that a meandric system is the following combinatorial structure:



Some basic questions about a uniform meandric system m_n of size n:

① How can I sample a uniform meandric system m_n?

② How many loops in m_n?

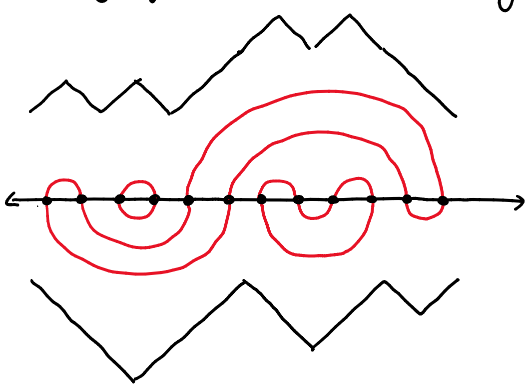
③ What is the size of the largest loop in m_n?

④ Is there typically a single loop of m_n which is much larger than the other loops? Or, are there multiple loops of comparable size?

⑤ Is there a sort of scaling limit for m_n?

① How can I sample a uniform meandric system m_n?

This question has a (surprisingly) simple answer (note that this is not the case for meanders). Indeed, we have the following bijection between meandric systems & pairs of walk excursions:



Then, in order to sample a uniform meandric system of size n, it is enough to sample an independent pair of walk excursions of time duration $2n$ and then apply the bijection above.

One might now claim that since every question on the meandric system can be rephrased as a question on an independent pair of walk excursions of time duration $2n$, then most of the questions above should be "easy".

We will see that this is NOT the case! Why? Loops are a very complicated functional of the two walks, which is quite hard to study.

② How many loops in m_n?

The result was conjectured by Golden, Nica, Puder, 2020

This question has been solved by Féray & Thévenin (2022):

Theorem: There exists a constant $c \in (0,1)$ such that

$$\frac{\#\text{ loops in } m_n}{n} \xrightarrow[n\to\infty]{\mathbb{P}} c$$

Moreover, $0.207 \leq c \leq 0.292$. [Numerical symulations: 0,23]

The key lemma for the proof is the following one:

Lemma: Let i_n be a uniform number in $\{1,\dots,2n\}$, then

$$\frac{\#\text{ loops in } m_n}{n} = \mathbb{E}\left[\frac{2}{|\ell_{i_n}(m_n)|} \,\middle|\, m_n\right],$$ where $|\ell_{i_n}(m_n)|$ denotes the size of the loop in m_n containing i_n.

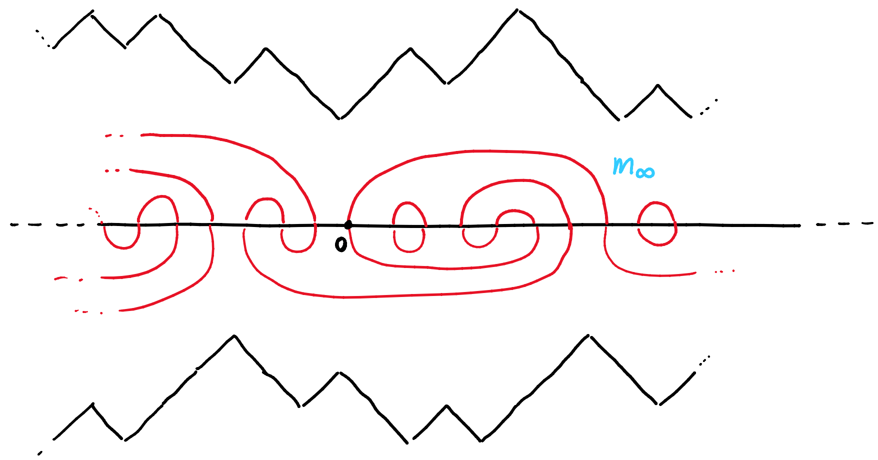
Proof: $$\mathbb{E}\left[\frac{2}{|\ell_{i_n}(m_n)|} \,\middle|\, m_n\right] = \sum_{i=1}^{2n} \frac{1}{\cancel{2}n}\,\underbrace{\frac{\cancel{2}}{\ell_{i_n}(m_n)}}_{\substack{\text{Each loop } \ell \text{ in } m_n \text{ contributes} \\ \frac{1}{|\ell|}\times|\ell| = 1 \text{ to the sum.}}} = \frac{\#\text{ loops in } m_n}{n}$$ $\square$

The rest of the proof of the theorem amounts in proving that

$$\mathbb{E}\left[\frac{2}{\ell_{in}(m_n)} \,\Big|\, m_n\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[\frac{2}{|\ell_0(m_\infty)|}\right] =: c$$

where m_∞ is the (quenched) Benjamini-Schramm local limit of m_n.
In particular, m_∞ is shown to be the uniform infinite meandric system (UIMS) m_∞ introduced by Curien, Kozma, Sidoravicius, Tournier (2017). $\rightsquigarrow$ We will come back to this paper

m_∞ is just the infinite meandric system obtained by replacing the two walk excursions above with two bi-infinite simple random walks:



Note that, in particular, we get that

$$c = \mathbb{E}\left[\frac{2}{|\ell_0(m_\infty)|}\right] = \sum_{n=1}^{\infty} \frac{\cancel{2}}{\cancel{2}n} \sum_{m\in\mathcal{M}_n} \overbrace{\mathbb{P}\big(\mathrm{Shape}(\ell_0(m_\infty))=m\big)}^{:=P_m}.$$

$\mathcal{M}_n$: meanders of size n

Moreover, these probabilities P_m can be explicitly computed in certain simple cases:

$$P_{\ominus} = \frac{2}{\pi} - \frac{1}{2} \approx 0.137$$

$$P_{\text{⦻}} = \frac{1}{4} - \frac{2}{3\pi} \approx 0.038$$

Open question: Is it true that $P_m \in \mathbb{Q}\left[\frac{1}{\pi}\right]$, $\forall m\in\mathcal{M}$?

③ What is the size of the largest loop?

Kargin (2020) shows that $\exists\, c>0$ s.t. w.h.p. (i.e. with probability one when $n\to\infty$):

$$\text{Size of the largest loop} \geq c\cdot \log(n).$$

His numerical simulations suggests that

$$\text{Size of the largest loop} \approx n^{\alpha} \quad \text{with} \quad \alpha = 4/5.$$

(Our simulations gives $\alpha \approx 0.792\ldots$)

④ Is there typically a single loop of m_n which is much larger than the other loops? Or, are there multiple loops of comparable size?

This question is closely related to the following question asked by Curien, Kozma, Sidoravicius, Tournier (2017):

Is there an infinite component in m_∞? (They call it the "INFINITE NOODLE")

Theorem:

$$\mathbb{P}(\#\text{ of infinite components of } m_\infty \in \{0,1\}) = 1 \quad \& \quad \mathbb{P}(\text{There is no INFINITE NOODLE}) \in \{0,1\}.$$

Conj: $\mathbb{P}(\text{There is no INFINITE NOODLE}) = 1$. [But without any specific motivation]

⑤ Is there a sort of scaling limit for m_n?

To the best of my knowledge nothing is known/conjectured before our work

3.3 A conjectural scaling limit

We can look at a meandric system m_n as a triplet

$$(M_n, \; P_n, \; \Gamma_n)$$

PLANAR MAP: (M_n)

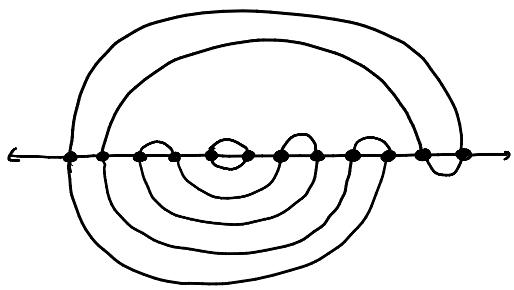
VERTICES = $\{1, \dots, 2n\}$

EDGES = $\left\{ \begin{array}{l} \text{arcs of meandric systems} \\ + \\ \text{pieces of the line between consecutive vertices} \\ + \\ \text{one additional edge from } 2n \text{ to } 1 \end{array} \right\}$

HAMILTONIAN CYCLE (P_n): determined by the line

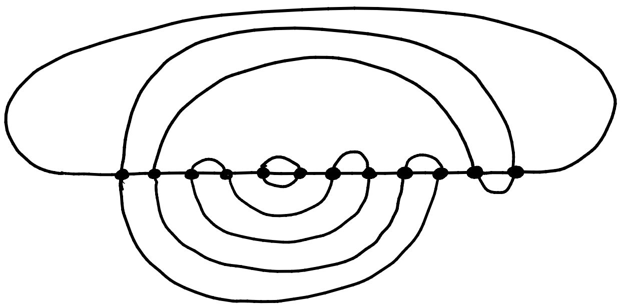
COLLECTION OF LOOPS (Γ_n): determined by the arcs

Example: If we consider this meandric system:

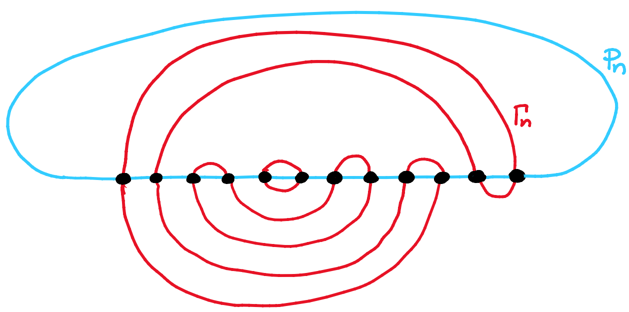


then

M_n =



and



Conjecture: (B., Gwynne, Park '23)

(M_n, P_n, Π_n) converges under an appropriate scaling limit to an independent triplet consisting of

$$(\sqrt{2}\text{-LQG sphere}, \ SLE_8, \ CLE_6)$$

What are these objects?

- The Liouville quantum gravity (γ-LQG) sphere with parameter $\gamma \in (0,2]$ is a random fractal surface with the topology of the sphere (it can be described by a random metric and a random measure on the Riemann sphere $\mathbb{C} \cup \{\infty\}$).

- The whole-plane Schramm-Loewner evolution (SLE_κ) with parameter $\kappa \geq 8$ is a space-filling (but non self-crossing) random fractal curve on $\mathbb{C} \cup \{\infty\}$.

- The whole-plane conformal loop ensemble (CLE_κ) with $\kappa \in (8/3, 8)$ is a random collection of loops which do not cross themselves or each other and wich locally look like SLE_κ.

Some simulations (which are important to get the correct intuition):

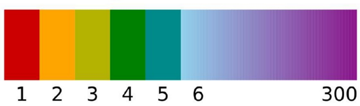
every vertex is the baricenter of the neighboors

Here we embedded the meandric sistem m_n in the sphere using the Tutte embedding:

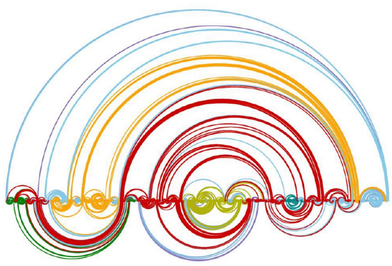
Planar map + loops

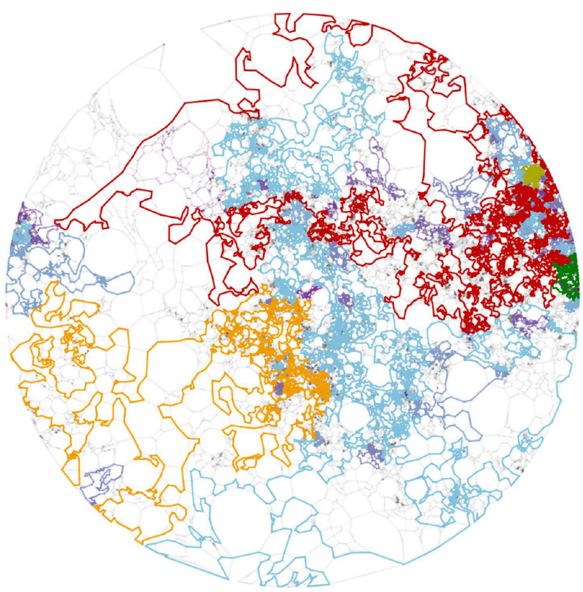
↓

√2-LQG + CLE6

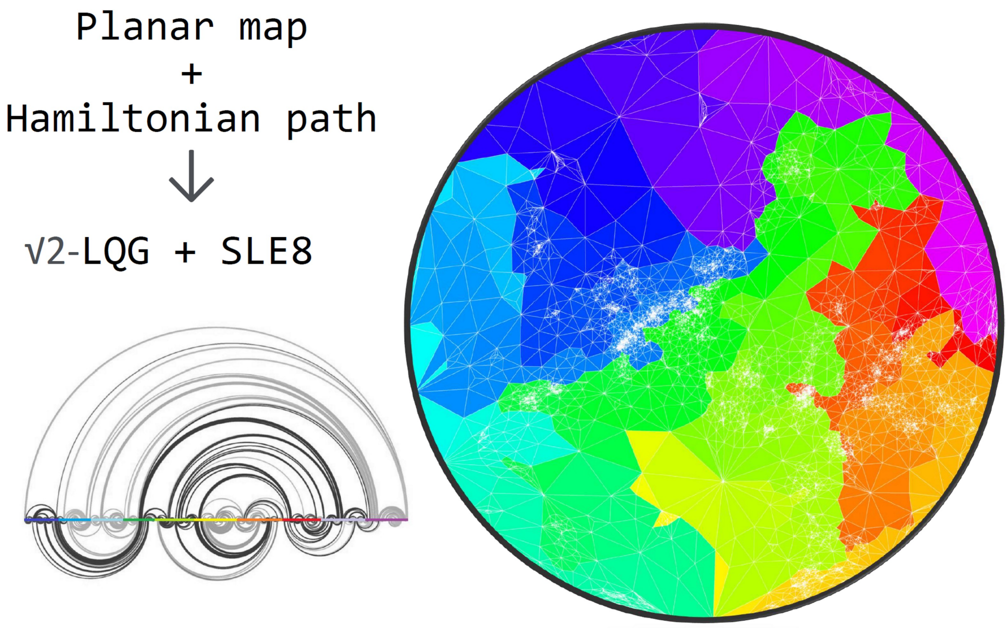


↳ The largest loop is in red

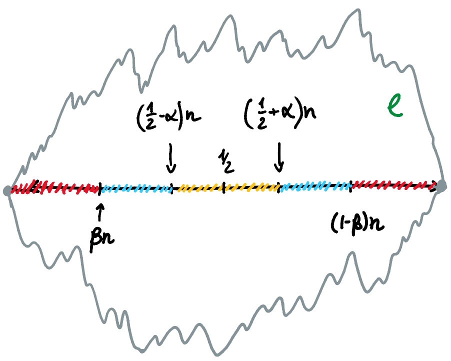




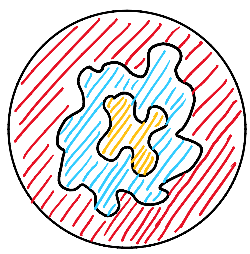
Here we embedded another meandric system m_n in the sphere and we color each face of the embedded dual map using the same color of the corresponding vertex in m_n (see the left-hand side):



<u>One important fact is the following one</u>: When we embed the meandric system we can find $\alpha > 0$ and $\beta > 0$ such that:



After embedding in the sphere we see the following in the limit:



All the 3 regions (red, blue, yellow) are <u>macroscopic</u> & red and yellow are <u>separated</u> by blue.

<u>Some remarks</u>:

- The conjecture is based on a combination of rigorous results, physics heuristics, and numerical simulations on a much more general model called

$$O(n \times m)\text{-loop model on planar maps.}$$

See

× "Permutons, meanders, & SLE-decorated Liouville quantum gravity". Borga, Gwynne, Sun.
↳ Section 6

× "On the geometry of uniform meandric systems". Borga, Gwynne, Park.
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for more explanations.

- The type of convergence should be quite general: for instance w.r.t. Gromov-Hausdorff topology for metric spaces or w.r.t. some (nice) embedding of M_n into $\mathbb{C} \cup \{\infty\}$.

- The 3 limiting objects are all independent (this is NOT true at the discrete level).
- $(\sqrt{2}\text{-LQG}, SLE_8)$ is the same limit as uniform planar maps decorated by a spanning tree.
- CLE_6 is the limit of the boundaries of clusters in Bernoulli percolation on the exagonal lattice
- We know how to prove that $(M_n, P_n) \longrightarrow (\sqrt{2}\text{-LQG}, SLE_8)$ at least in the Peano-sphere sense [this is a weak type of convergence but it is VERY HELPFULL for our results, see later...]
- The really hard part of the conjecture is to prove that $\Gamma_n \longrightarrow CLE_6$.
  One aspect that might explain why this is hard is the following fact:
  If (e^+, e^-) are the two limiting Brownian excursions for the pair of walk excursions determining m_n, then

  $$(e^+, e^-) \quad \text{and} \quad (\sqrt{2}\text{-LQG}, SLE_8)$$

  determines each other (in the sense that one is a meas. function of the other one) but CLE_6 is not determined by (e^+, e^-).
- This conjecture gives several new (conjectural) answers to the questions above. In particular:

③ What is the size of the largest loop?

④ Is there typically a single loop of m_n which is much larger that the other loops? Or, are there multiple loops of comparable size?

Conjecture:

$$\#\{\text{vertices of the } k\text{-th largest loop of } m_n\} = n^{\alpha + o(1)}$$

$$\text{where} \quad \alpha = \underbrace{\tfrac{1}{2}(3-\sqrt{2})}_{\substack{0.7929\\ \text{SS}}} = \underbrace{\text{Hausdorff dimension of } CLE_6 \text{ w.r.t. } \sqrt{2}\text{-LQG metric.}}_{\text{Computed using the KPZ relation}}$$

In particular, there should be no infinite noodle but many macroscopic loops.