3.4 - Rigorous results on meandric systems

Let

$$d = d_{\sqrt{2}} := \text{Hausdorff dimension of the } \sqrt{2}\text{-LQG sphere}$$

[Intuitively this means that a Ball of radius r has measure $\approx r^d$]

The number d exists and is a.s. constant thanks to [Gwynne, Pfeffer, '22]. The exact value is <u>NOT</u> known (this is one of the most important open problems in the field) but [Gwynne, Pfeffer, '19] proves that

$$3.55 \approx 2(9+3\sqrt{5}-\sqrt{3})(4-\sqrt{15}) \leq d \leq \frac{2}{3}(3+\sqrt{6}) \approx 3.63.$$

Moreover, the fact that $(M_n, P_n) \longrightarrow (\sqrt{2}\text{-LQG}, SLE_8)$ in the Peano sphere sense, gives that

Proposition: W.h.p. $\text{diameter}(M_n) = n^{1/d + o(1)}$ (w.r.t. the graph distance in M_n), that is, $\forall\, \xi \in (0,1)$

$$\mathbb{P}\left(n^{1/d - \xi} \leq \text{diameter}(M_n) \leq n^{1/d + \xi} \right) \xrightarrow[n \to \infty]{} 1$$

The proposition is proved using standard techniques and follows from an important result of [Gwynne, Holden, Sun, '20] that allows to compare distances between planar maps and the limiting LQG-surface when one has convergence in the Peano-sphere sense.

We are finally ready to state our main results:

THEOREM 1 (B., Gwynne, Park '23)

W.h.p., $\exists\ \ell \in \Gamma_n$ such that $\text{diameter}(\ell) \geq n^{1/d - o(1)}$

Remarks:

- The theorem above proves (modulo the $o(1)$ correction in the exponent) that loops in Γ_n do <u>not</u> collapse to points in the limit, i.e. there are macroscopic loops.
- The $o(1)$ term is there because there are no available techniques to get up-to-constants bounds for graph-distances in M_n.
- $\text{diameter}(\ell) \geq n^{1/d - o(1)} \implies \#\{\text{vertices in } \ell\} \geq n^{1/d - o(1)}$

- $\text{diameter}(\ell) \geq n^{1/d - o(1)} \implies \#\{\text{vertices in } \ell\} \geq n^{1/d - o(1)}$

- $0.275 \leq 1/d \leq 0.282$ from the previous bounds on d.

- This is the first power-law bound (before Kargin proved $\log(n)$).

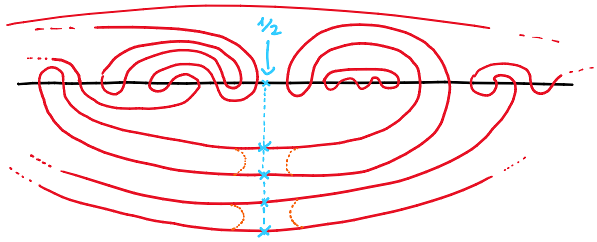
- $n^{1/d}$ is quite far from the previously conjecture n^{α}, $\alpha \approx 0.79$:
  In order to get this improvement, one needs to show that the loop ℓ is much longer that a geodesic in M_n (which requires a much finer understanding of the geometry of ℓ w.r.t. M_n).

- There are two recent works:
  - Duminil-Copin, Glazman, Peled, Spinka, 21
  - Crawford, Glazman, Harel, Peled, 20

  which prove existence of macroscopic loops for the critical $O(n)$-loop model on the hexagonal lattice in two different ranges for the parameter n.
  The results in these works and our results are quite similar in spirit, but the proofs are completely different (we are working on a random lattice).

THEOREM 2 (B., Gwynne, Park, '23)

Consider the infinite meandric system m_∞:



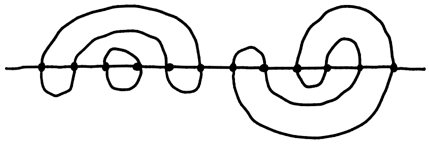
Cut all the arcs crossing the blue ray above (there are infinitely many arcs to cut) and rewire successive pairs of unmatched ends as shown in the picture (orange dotted lines).
Then, a.s., there are NO infinite loops.

Remark: The new m.s. above is called "Uniform Infinite Half Plane Meandric System" (UIHPMS) and it is conjectured to have exactly the same scaling limit of a uniform m.s. but with the topology of the half plane (instead of the sphere).

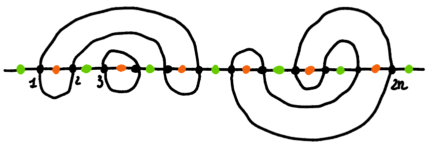
Remark: Morally, this is equivalent to say that there is no infinite cluster in percolation after removing a ray from the origin in the plane.

3.5 Meandric systems & percolation

Consider a meandric system:



Color the half integers alternatively in green and orange:

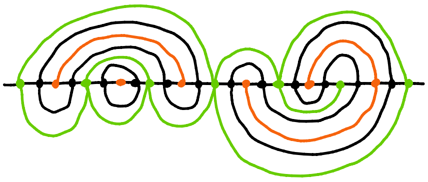


Note that

green $= x - \frac{1}{2}$ with $x \in \mathbb{Z}$ odd

orange $= x - \frac{1}{2}$ with $x \in \mathbb{Z}$ even

Now, if possible, match each green (resp. orange) vertex with the closest green (resp. orange) vertex on the left and on the right that is matchable without crossing black arcs:



Note that each orange/green vertex has 4 potential half-arcs:

Claim:

i.e. it determines the m.s.

① The configuration of green (resp. orange) arcs determines the black arcs and the orange (resp. green) arcs.

② If for each green vertex we independently sample each of the four potential half-arcs uniformly and indep. at random, then we match half-arcs into arcs in the unique possible planar way, & finally we determine the black arcs configuration as in ①, then we get exactly a UIMS M_∞.

Proof: Exercise.

In particular, we have the following percolation point-of-view:

green as open edges / orange as closed edges / black as boundaries of clusters.

3.6 - Proof of THEOREM 1 (Existence of macroscopic loops)

The proof of THEOREM 1 is divided into two main steps:

- STEP 1: Show that $\exists$ a loop in Γ_n which is big w.r.t. the metric induced by Γ_n.
- STEP 2: SLE/LQG argument to lower-bound distances in Γ_n.

  $\hookrightarrow$ Here we use some fundamental results:

  - Existence of the LQG-metric (Ding-Dubedat-Dunlap-Falconet + Gwynne, Miller)
  - Mating of trees (Sheffield-Miller-Duplantier)

I will only explain STEP 1. STEP 2 is quite standard (but you need to know quite a lot about SLEs and LQG).

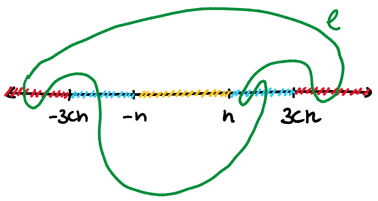
The main result in STEP 1 is the following theorem about the infinite m.s. m_∞:

Theorem: For n large enough, the following is true:

For every constant $C>1$, at least one of the following events holds with prob. $\geq \frac{1}{10}$:

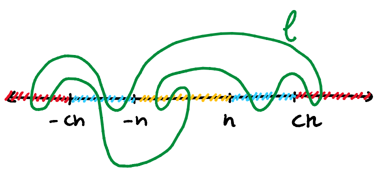
(A) $\exists$ loop ℓ such that:

- $[-n,n]$ is disconnected from ∞ by ℓ
- ℓ hits $[-3Cn,-n) \cup (n,3Cn]$



(B) $\exists$ loop ℓ (or an infinite path ℓ) such that:

- ℓ hits $[-n,n]$
- ℓ hits $\mathbb{Z} \setminus [-Cn,Cn]$



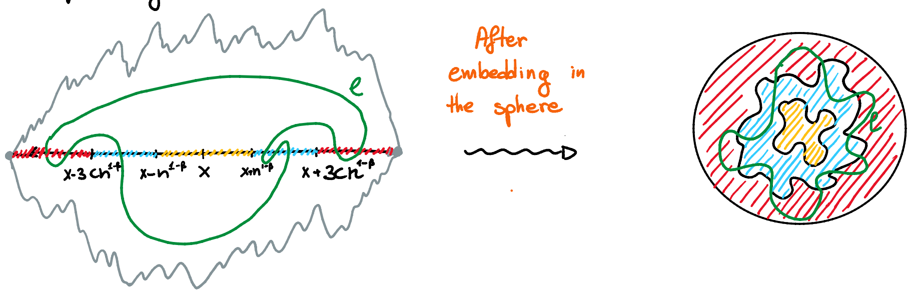
INTUITIVELY: Why is this enough to conclude that there are MACROSCOPIC LOOPS?

SKETCH: Using the independence between the event above and the same event translated to the right, we can ensure that the event above (with $n^{1-\beta}$ in place of n) happens somewhere in $[0,2n]$ with probability

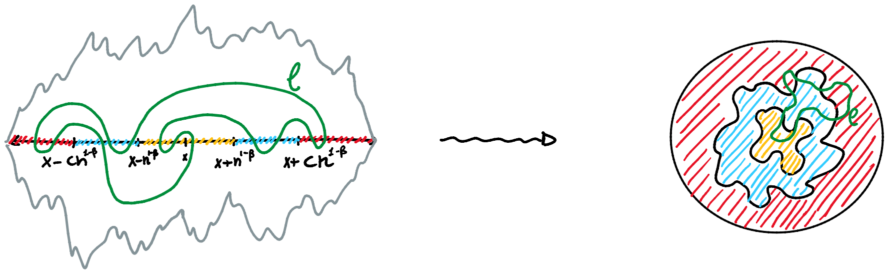
$$1-a_0 e^{-a_1 n^\beta}$$

for some $a_0, a_1, \beta > 0$. Then we look at this place where the event holds and we conclude as follows: We condition on the event E_n that the two walks take an excursion between 0 and $2n$ (this event has probability $\approx a_3 \cdot n^{-p}$, for some $p>0$). Conditioning on E_n, m_∞ between 0 and $2n$ is a uniform meandric system of size n.

Hence with conditional probability at least $1 - a_0 \cdot a_3^{-1} \cdot n^{\beta} \cdot e^{-a_1 n^{\beta}}$ given E_n we get that one of the following two events hold:



QUESTION: Why do we also need to ensure that ℓ intersects the blue region?
ANSWER: Because we do not want that the loop becomes tiny in the other side of the sphere!



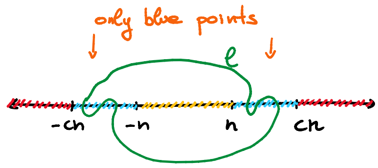
If C is big enough, one can show that the 3 regions (red, blue, yellow) are all macroscopic in the limit and so conclude that ℓ is also macroscopic (in the first case ℓ must surround the yellow region and cannot become tiny in the other side of the sprere; while in the second case ℓ must traverse the blue region).

Proof of the theorem:

Note that if $\exists$ an infinite loop in m_∞ then the theorem is trivially true. So for the rest of the proof we are going to assume that a.s. m_∞ has no infinite path.

We consider the following event:

$E_n := \{ \exists$ loop ℓ such that

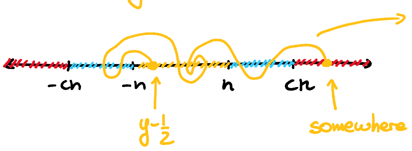


$\}$

If $\mathbb{P}(E_n) \geq \frac{1}{10}$ then we are done since (A) holds. Hence we assume that

$$\mathbb{P}(E_n^c) > 1 - \frac{1}{10} \quad \& \quad E_n^c \text{ holds}$$

If E_n^c holds then $\exists$ $y \in [-n+1, n]$ such that



This is either a sequence of green arcs (if y odd)
or a " " orange " (if y even)

somewhere in $\mathbb{Z} \setminus [-3cn, 3cn]$

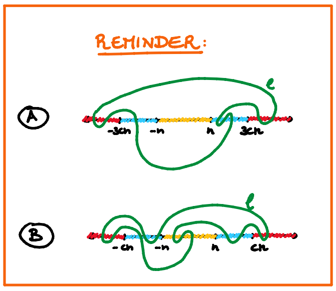
Proposition: With probability $\geq \frac{1}{10}$, $\exists$ loop such that:

1. ℓ disconnects $y-\frac{1}{2}$ from ∞ for some $y \in [-n+1, n]$
2. ℓ hits a point in $\mathbb{Z} \setminus [-Cn, Cn]$
3. ℓ hits a point of $[y, 3Cn]$

Why is this enough to conclude?

- If ℓ disconnects $[-n,n]$ from ∞ then ③ $\Longrightarrow$ Ⓐ
- $\underbrace{\text{If not}}$, then ① + ② $\Longrightarrow$ Ⓑ

Indeed these two conditions together imposes that ℓ must hit a point in $[-n,n]$

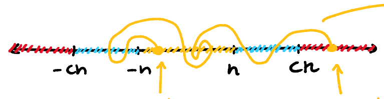


Proof of the Proposition:

Recall that we are assuming that

$$\mathbb{P}(E_n^c) > 1 - \frac{1}{10} \quad \& \quad E_n^c \text{ holds}$$

and if E_n^c holds then $\exists$ $y \in [-n+1, n]$ such that



This is either a sequence of green arcs (if y odd)
or a " " orange " (if y even)

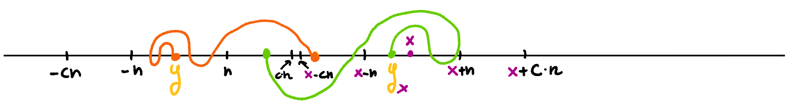
$y-\frac{1}{2}$ somewhere in $\mathbb{Z} \setminus [-3Cn, 3Cn]$

Now w.l.o.g. we can assume that $\mathbb{P}(y \text{ even} \mid E_n^c) \geq \frac{1}{2}$.

As a consequence, we can take $x \in [2Cn+1, 2Cn+3]$ odd such that:

$$\mathbb{P}\Big(\underbrace{E_n^c \cap \{y \text{ even}\}} \cap \underbrace{E_n^c(x) \cap \{y_x \text{ odd}\}} \Big) \geq \frac{\mathbb{P}(E_n^c)^2}{4}$$

independent



y is orange and y_x is green $\Longrightarrow$ $\exists$ loop ℓ separating y and y_x.

- Then such loop ℓ must hit a vertex in $[y, y_x] \subseteq [y, 3Cn]$ $\Longrightarrow$ ③ holds.

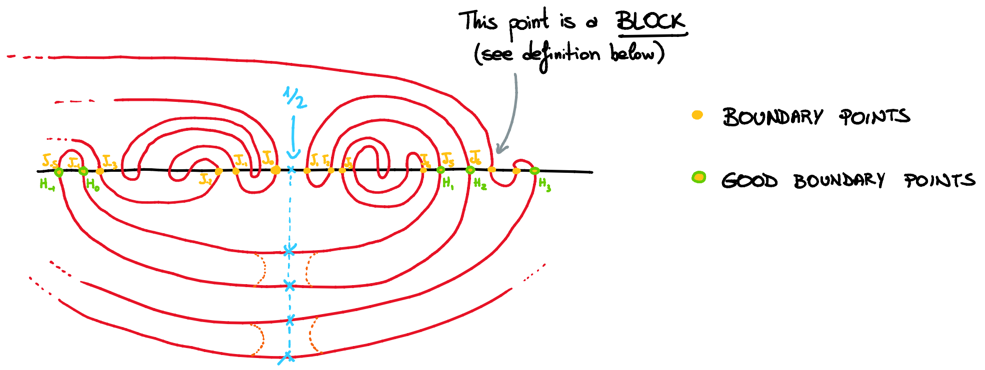
- Moreover, with probability at least $\frac{\mathbb{P}(E_n^c)^2}{8} \geq \frac{1}{10}$, ℓ disconnects $y - \frac{1}{2}$ from infinity

$\Longrightarrow$ ① and ② hold. $\square$

↳ because E_n^c holds.

3.7 Proof of THEOREM 2

Recall the UIHPMS:



Def: A point $x \in \mathbb{Z}$ is called a BOUNDARY POINT if it can be connected to the blue ray $\frac{1}{2} \times (-\infty, 0]$ by some continuous path without crossing any arc or the real line.
A point $x \in \mathbb{Z}$ is called a GOOD BOUNDARY POINT if it is a boundary point and is also an end-point of the rewired arcs.

We denote the boundary points by J_k, $k \in \mathbb{Z}$ so that $J_0 = 0$ (see picture above).
" " " good bound. points " H_m, $m \in \mathbb{Z}$ so that $0 \in [H_0, H_1 - 1]$ (see picture above).

Observation 1: Note that if $x = J_{2k} \in 2\mathbb{Z}$ is a boundary point then $x+1 = J_{2k+1} \in 2\mathbb{Z}+1$ is also a boundary point.
As a consequence, the law of the UIHPMS is invariant under J_{2k}-translations, for all $k \in \mathbb{Z}$.

Observation 2: The fact that a boundary point is a good boundary point is independent of the UIHPMS (see our paper for a careful explanation of this fact.)

Definition: For each good boundary point H_m, let P_m be the unique directed path of arcs in the UIHPMS starting from H_m, following the arc incident to H_m in the upper half-plane and ending at the first good boundary point other than H_m (if it exists), otherwise let P_m be the whole semi infinite path started from H_m.
We call P_m the BOUNDARY PATH started from H_m.

Lemma: Almost surely, the boundary path P_m is finite, $\forall m \in \mathbb{Z}$.

To prove the lemma we need one more definition:

Def: We call $x \in 2\mathbb{Z}_{>0}$ a UPPER BLOCK if

x is linked with an upper-arc to a point in $(-\infty, 0]$.

If, in addition, $x = J_{2k}$ for some $k \in \mathbb{Z}$, we call x a BLOCK.

See the point J_6 in the figure above for an example.

Fact: Fix $\varepsilon > 0$. Almost surely, there are infinitely many $k > 0$ such that J_{2k} is a block and there are at most $\varepsilon \cdot k$ upper blocks in $(0, J_{2k}]$.

Brief explanation: One can show using some random walk estimates that $\mathbb{P}(A_n) \geq \frac{c}{n}$, where

$A_n = \{ 2n = J_{2k}$ for some $k \geq \frac{\sqrt{n}}{2}$ & there are at most $\varepsilon \cdot \sqrt{n}$ upper blocks in $(0, J_{2k}] \}$

Then one concludes that $P(A_n \text{ i.o.}) > 0$ by Kochen-Stone theorem. Finally $P(A_n \text{ i.o.}) = 1$ by 0-1-laws.

Proof of the Lemma:

We first look at boundary paths P_m started at good boundary points of the form J_{2k}, for some $k \in \mathbb{Z}$. (Recall that the UIHPMS is invariant under J_{2k}-translations)

We set

$$E_k := \{ J_{2k} = H_m \text{ for some } m \in \mathbb{Z} \ \& \ P_m \text{ is semi-infinite} \}$$

Note that $\mathbb{P}(E_k) = p$ for all $k \in \mathbb{Z}$ (translation invariance + J_{2k} is good is indep. of the UIHPMS)

Assume for a contradiction that $p > 0$. By the Birkhoff ergodic theorem:

$$\lim_{k \to \infty} \frac{1}{k} \# \{ k' \in [1,k] \cap \mathbb{Z} : E_{k'} \text{ occurs} \} = p \quad \text{almost surely.}$$

By this & the Fact above, we have that almost surely, there exist arbitrarily large $k \in \mathbb{Z}_{>0}$ s.t.:

from Fact with $\varepsilon = p/4$:

(i) J_{2k} is a block;

(ii) there are at most $p/4 \cdot k$ upper blocks in $(0, J_{2k}) \cap \mathbb{Z}$;

(iii) $\# \{ k' \in [1,k] \cap \mathbb{Z} : E_{k'} \text{ occurs} \} > p/2 \cdot k$.

Fix $\kappa \in \mathbb{Z}_{>0}$ with the above 3 properties.

For $\kappa' \in (0,\kappa) \cap \mathbb{Z}$ such that $E_{\kappa'}$ occours, let $m(\kappa') \in \mathbb{Z}$ such that $H_{m(\kappa')} = J_{2\kappa'}$.

$P_{m(\kappa')}$ is semi-infinite, so it must exit $(0, J_{2\kappa})$ and it can do it only through an upper-arc connecting $(0, J_{2\kappa})$ to $(-\infty, 0]$ because $J_{2\kappa}$ is a block & there are no lower-arcs crossing $\frac{1}{2} \times (-\infty, 0]$.

But these upper arcs are then upper-blocks by definition. Hence we have at least $\frac{p}{2} \cdot \kappa$ upper blocks by (iii). A contradiction with (ii). Hence $p = 0$.

Finally, one repeats the same argument for points of the form $J_{2\kappa+1}$. $\square$

The last proved lemma gives a natural matching of the good boundary points $\{H_m\}_{m \in \mathbb{Z}}$

H_m is matched with $H_{m'}$ iff P_m ends at $H_{m'}$.

One can show that this matching is a <u>strongly ergodic random non-crossing perfect matching</u>.

The GOOD BOUNDARY POINTS are needed here!

The matching is translation invariant

Any event wich is invariant under even shifts has probability zero or one.

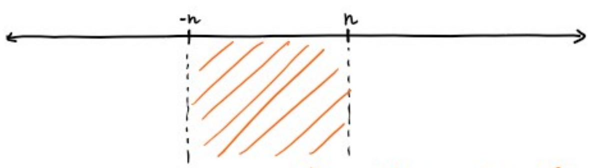
This implies (after some work) the following result:

Lemma: Almost surely, there exists infinitely many $m \in \mathbb{Z}_{>0}$ such that the boundary path P_m ends at $H_{m'}$ for some $m' \in \mathbb{Z}_{\leq 0}$.

We can now complete the proof of the THEOREM 2:

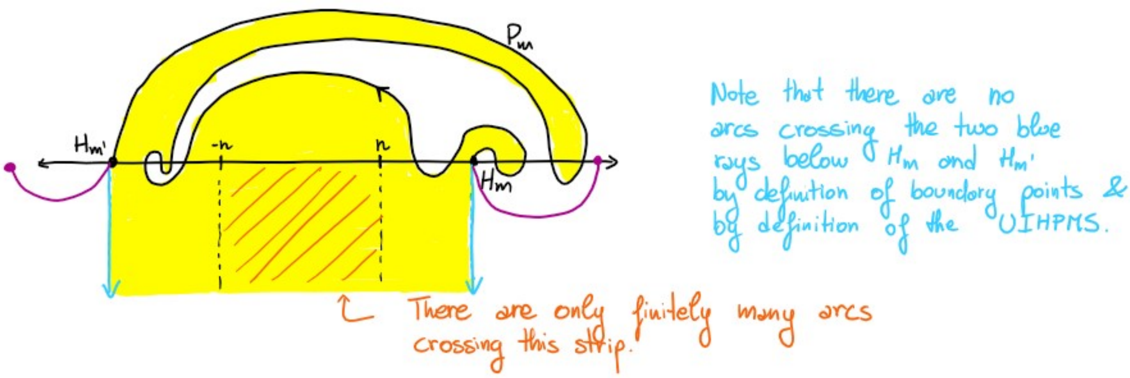
Fix $n \in \mathbb{N}$. We will show that a.s. there is no bi-infinite path of arcs in the UIHPMS which intersects $[-n,n]$.

Note that



There are only finitely many arcs crossing this strip.

By our lemma above, we can then find m large enough such that P_m ends at $H_{m'}$ for some $m' \in \mathbb{Z}_{<0}$ and P_m never cross $[-n,n]$:

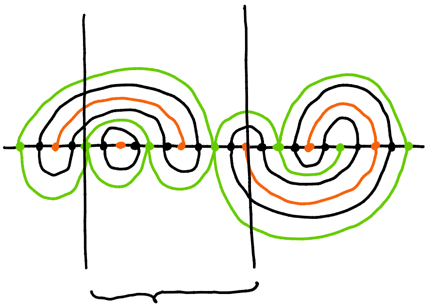


Now, note that a bi-infinite path needs to enter and exit the yellow region in order to cross $[-n,n]$ and this is possible only throug the lower arcs connected to H_m and $H_{m'}$ (see the two purple arcs). But then, by construction of P_m we would not touch $[-n,n]$. Hence there is no bi-infinite path crossing $[-n,n]$. $\square$

3.8 Open questions

Here are some further open problems on meandric systems that I like:

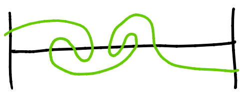
Given a m.s. one can consider the following boxes starting at green and ending at orange:



A box of size 5 (= number of black dots inside the box)

Note that each box has a top-left / top-right / bottom-left / bottom-right side.

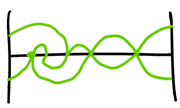
In our paper we showed that the probability that $\exists$ a top-left to bottom-right green crossing is:

$\mathbb{P}\Big($  $\Big) = \frac{1}{2}$

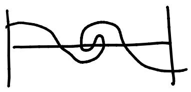
Question 1: Can we compute the same crossing event for loops instead of green-paths?

Question 2: Is there a good notion of FKG for this model?

Remark: Crossing events seems to be anti-correlated, that is:

$\mathbb{P}\Big($  $\Big) \leqslant \frac{1}{4}$

I guess this would immediately gives the following lower bound for Question 1:

$\mathbb{P}\Big($  $\Big) \geqslant \frac{1}{4}$