4 The meandric permuton: A new scaling limit for uniform meanders

4.1 - Permutons

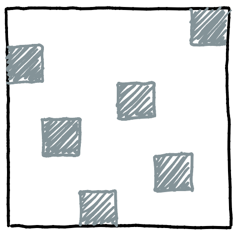
Definition: A PERMUTON μ is a probability measure on the unit square $[0,1]^2$ with uniform marginals, i.e. $\mu([a,b]\times[0,1]) = \mu([0,1]\times[a,b]) = b-a \quad \forall\, 0\le a\le b\le 1$.

There is a natural way to associate a permutation with a permuton.

Example: If

$$\sigma = 5\ 3\ 1\ 4\ 2\ 6 \qquad \text{(one-line notation)}$$

then

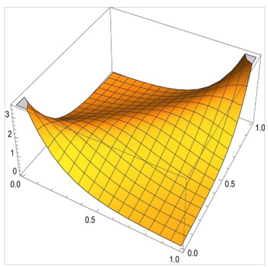
$\mu_\sigma =$  (Lebesgue measure on the gray support of total mass 1)

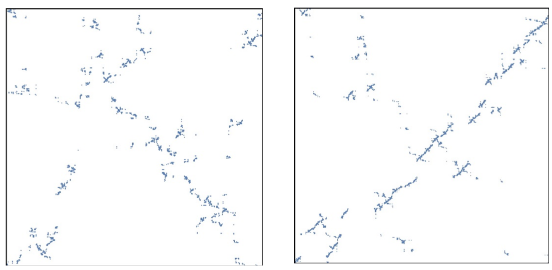
We then have a natural notion of convergence for permutations:

$$\sigma_n \xrightarrow{n\to\infty} \mu \iff \mu_{\sigma_n} \xrightarrow{n\to\infty} \mu \quad \text{w.r.t. weak-topology.}$$

Some interesting permutons:


Uniform


Mallows


Pattern-avoiding

A BIT OF HISTORY:

- Introduced by Hoppen, Kohayakawa, Moreira, Rath, Sampaio in 2011 to study permutation sequences and parameter testing (as a counterpart of graphons for graph).

- Popularized in the probabilistic community by:

  * Kenyon, Kral, Radin, Winkler ⇝ Permutations with fixed pattern densities.
  * Bassino, Bouvel, Féray, Gerin, Maazoun, Pierrot ⇝ Limits of pattern-avoiding permutations

- There is now a quite large literature studying limits of non-uniform random permutations (see, for instance, my PHD thesis for an overview).

4.2 - An instructive example: The case of Baxter permutations

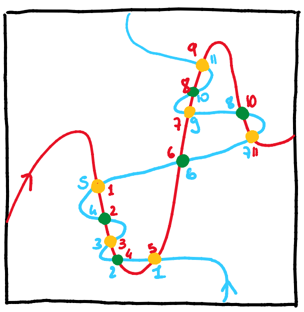
Def: A MONOTONE MEANDER of size n is a pair of simple curves ℓ_1 and ℓ_2 in $[0,1]^2$ which cross each other exactly $2n-1$ times and such that:

- ℓ_1 starts on the left-hand side of $[0,1]^2$, ends on the right-hand side of $[0,1]^2$ and never moves in the left direction. (Equivalently, it is the graph of a cont. function from $[0,1]$ to $[0,1]$.)

- ℓ_2 starts on the bottom side of $[0,1]^2$, ends on the top side of $[0,1]^2$ and never moves in the bottom direction. (Equivalently, it is the graph of a cont. function from $[0,1]$ to $[0,1]$ rotated by 90 degrees).

We identify two monoton meanders using the usual identification for meanders.

Example: There is a natural way to encode a monotone meander with a permutation:



$$\longrightarrow \quad \sigma = 5\ 4\ 3\ 2\ 1\ 6\ 9\ 10\ 11\ 8\ 7$$

φ

Reduce the yellow permutation to the unique permutation on $\{1,\dots,n\}$ preserving the relative order between numbers.

$$\pi = 3\ 2\ 1\ 5\ 6\ 4$$

"Def": The set φ (Monotone meanders) is called the set of Baxter permutations.

Proposition: The map φ is a bijection between motone meanders of size n and Baxter permutations of size n.

Baxter permutations enjoys several nice combinatorial properties & equivalent definitions.

Def: A BAXTER PERMUTATION is a permutation avoiding the patterns 2-41-3 and 3-14-2

i.e. σ is Baxter $\iff \nexists\ i_1<i_2<i_3$ s.t. $\sigma(i_2+1)<\sigma(i_1)<\sigma(i_3)<\sigma(i_2)$ or $\sigma(i_2)<\sigma(i_3)<\sigma(i_1)<\sigma(i_2+1)$

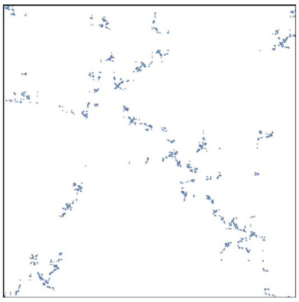
Theorem: (Borga, Maazoun, 2021)

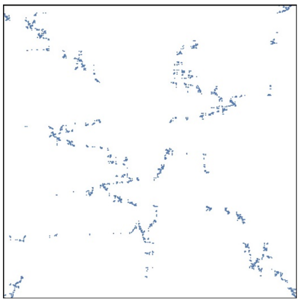
If σ_n is a uniform Baxter permutation of size n, then

$$\mu_{\sigma_n} \xrightarrow[n\to\infty]{d} \mu_B =: \text{Baxter permuton}$$

Remark: The Baxter permuton μ_B is a random permuton, i.e. a random probability measure of the unit square $[0,1]^2$.

Simulations:





Baxter permutations

IMPORTANT COMMENSTS:

- These permutations are in bijection with other interesting combinatorial objects, such as trees, walk in cones, bipolar orientations, tessellations, etc.
- The proof of the theorem above involves the study of certain coalescent-walk processes.

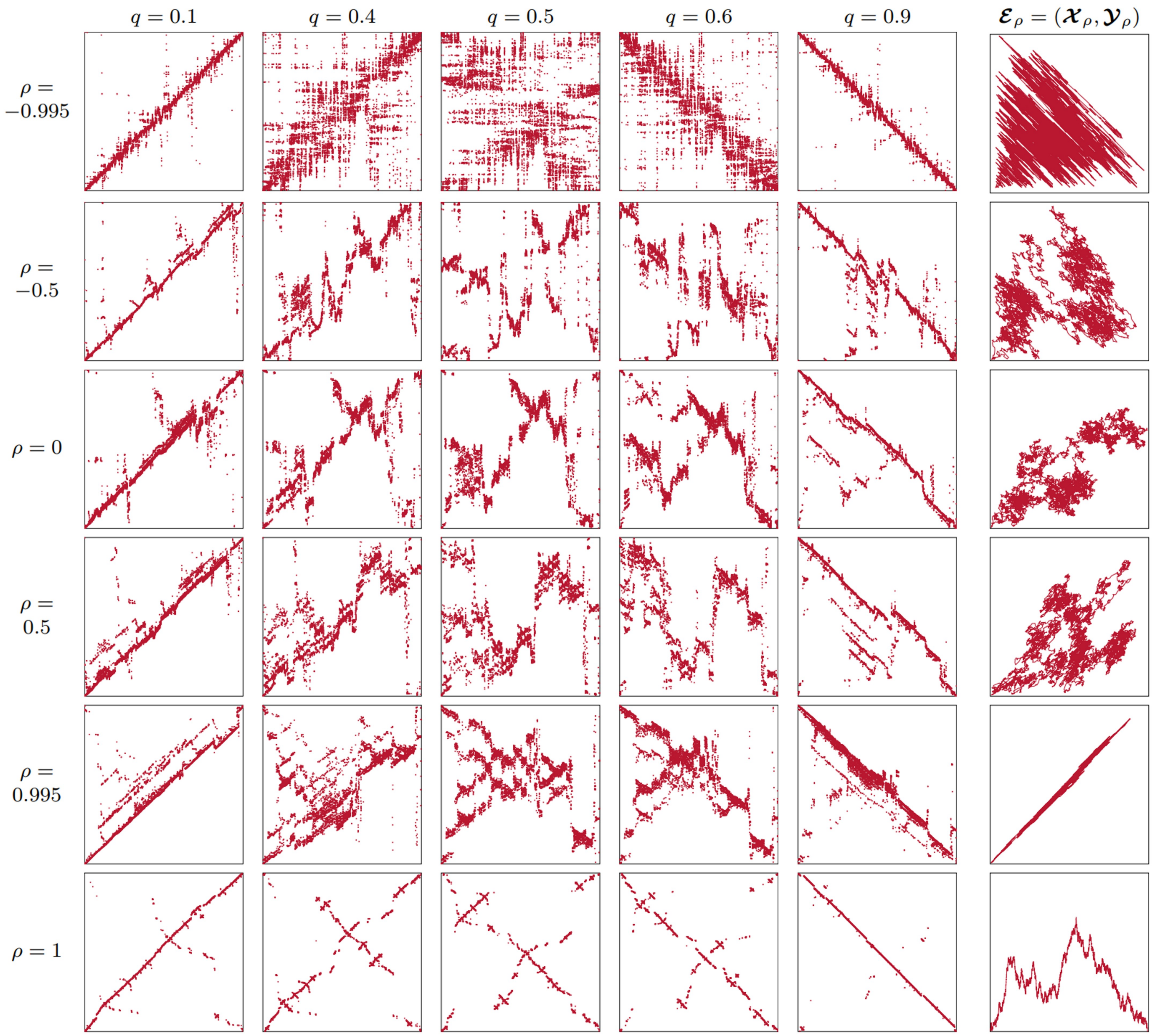
- The Baxter permuton can be constructed from a two-dim. correlated Brownian exc.
  +
  a system of SDEs.

- I recently introduced a two-parameter family $(\mu_{\rho,q})_{\rho\in[-1,1],\, q\in[0,1]}$ of random permutons, called skew Brownian permutons which are universal objects (i.e. scaling limits of many models of random permutations) and such that

$$\mu_{-\frac{1}{2},\frac{1}{2}} = \mu_B \quad \& \quad \mu_{1,q} = \text{Brownian separable permuton } (q) \quad \forall q\in[0,1]$$

↳ Another important family of random permutons.

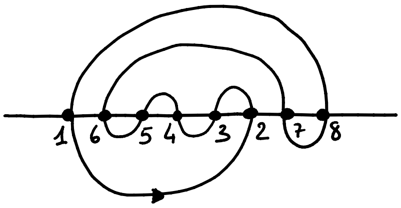
Simulations:



4.3 - The meandric permuton (PART 1)

We can encode a meander with its "meandric permutation" in a similar way:

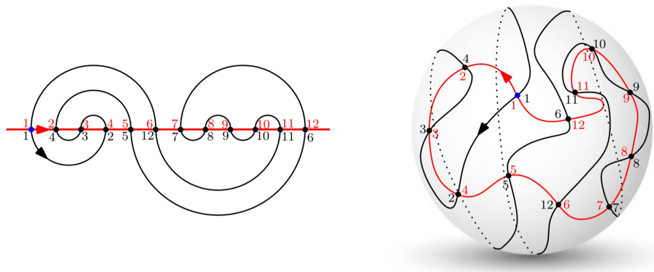
Example:

meander = 

$\xrightarrow{\varphi}$ meandric permutation = 1 6 5 4 3 2 7 8

Def: A MEANDRIC PERMUTATION of size $n \in \mathbb{N}$ is a permutation that can be obtained from the map φ.

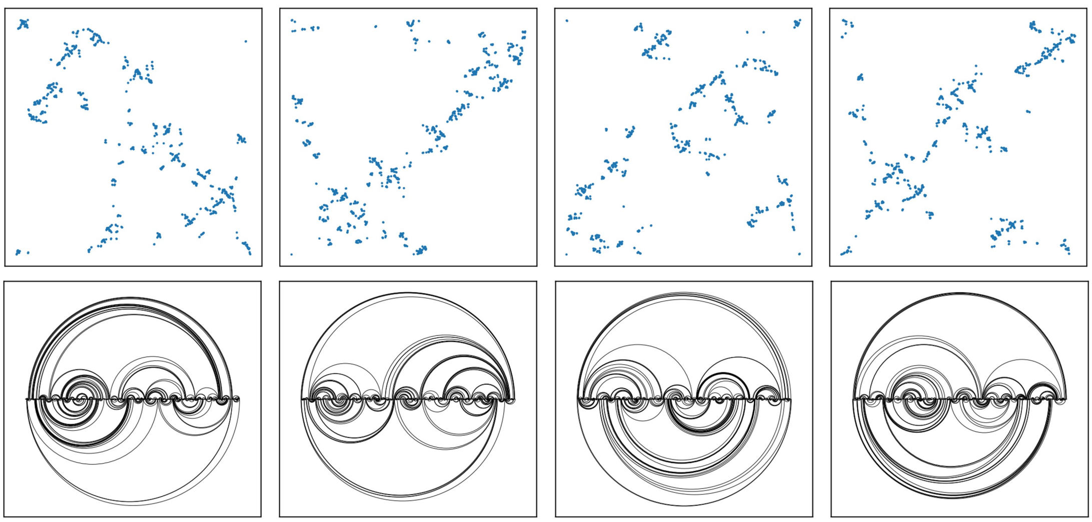
Sometime it is convenient to think about meanders on the sphere:



"Definition:" The MEANDRIC PERMUTON is the conjectural scaling limit of uniform meandric permutations.

Later we will propose an explicit construction of the meandric permuton.

Simulations:



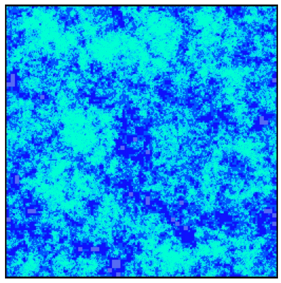
4.4 - Some important objects in Random geometry: SLEs & LQG

We only give informal definitions here:

Definition: For $\gamma \in (0,2)$, a *γ-LQG area measure* μ is a random probability measure on $\mathbb{C}$ obtained as the limit of regularized versions of

$$e^{\gamma \cdot h}\, d^2 z, \quad \text{where} \quad \begin{cases} h \text{ is a whole-plane GFF} \\ d^2 z \text{ is Lebesgue-measure on } \mathbb{C}. \end{cases}$$

[FOR EXPERTS: μ is the area-measure corresponding to a singly-marked unit-area γ-quantum sphere.]



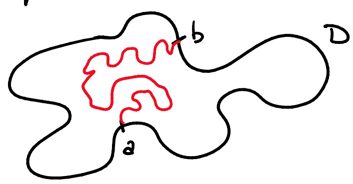
IMPORTANT FACTS about μ:

- μ is random;
- $\mu(\mathbb{C}) = 1$ a.s.
- μ is a.s. non-atomic;
- μ a.s. assigns positive mass to open subsets.

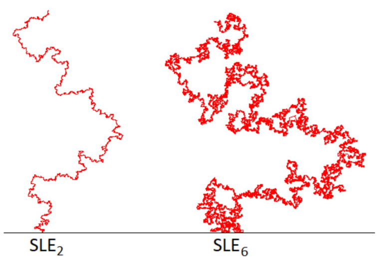
Definition: Let D be a simply connected, open, complex domain not equal to $\mathbb{C}$. Fix $a, b \in \partial D$. A *CHORDAL SLE from a to b in D* is a one-parameter family (indexed by $\kappa \geq 0$) of random non-crossing curves (viewed modulo time parametrization) in D from a to b wich satisfies:

× conformal invariance,

× a certain domain Markov property.



IMPORTANT FACTS about CHORDAL SLEs:

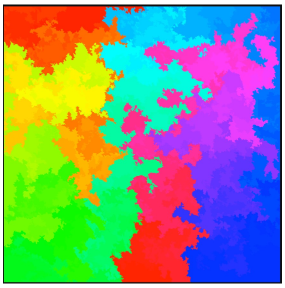


- These curves are non-crossing
- For $0 \leq \kappa \leq 4$ the curve is a.s. simple
- For $4 < \kappa < 8$ the curve a.s. touches itself & every point is contained in a loop
- For $\kappa \geq 8$ the curve is a.s space-filling.

Definition: The whole-plane space-filling $SLE_{\kappa'}$ (from ∞ to ∞) is a random space-filling curve η in $\mathbb{C}$ which:

- starts and ends at ∞;
- when $\kappa' > 8$, η is just a two-sided variant of chordal $SLE_{\kappa'}$.
- when $\kappa' \in (4,8)$, η can be obtained from a two-sided variant of $SLE_{\kappa'}$ by iteratively "filling-in" the loops which it disconects from its target point.

IMPORTANT FACTS about whole-plane SLEs:



- These curves are non-crossing but space-filling for all $\kappa' > 4$.
- These curves are invariant under scaling, translation, rotation & time-reversal.
- Given a γ-LQG μ and an independent whole-plane $SLE_{\kappa'}$ we always parametrize η with μ, i.e.

$$\mu\left(\eta[0,t]\right) = t, \quad \forall\, t \in [0,1].$$

4.5 - Constructing permutons from SLEs and LQG: The meandric permuton (PART 2)

The recipe:

Fix $\gamma \in (0,2)$ and $\kappa_1, \kappa_2 > 4$.

Let

INDEPENDENT

- μ be a γ-LQG area measure
- (η_1, η_2) a pair of whole-plane space-filling SLEs of parameters (κ_1, κ_2).

↳ The coupling between η_1 and η_2 is NOT specified for the moment.

Then

1) We parametrize η_1 and η_2 with μ;

2) We consider the function $\psi : [0,1] \longrightarrow [0,1]$ such that $\eta_1(t) = \eta_2(\psi(t))$, for all $t \in [0,1]$.

↳ Informally, the "continuum permutation" obtained by comparing the order in which η_1 and η_2 hits the points of $\mathbb{C}$.

3) We define the permuton associated with (μ, η_1, η_2) by

$$\pi(A) := \underbrace{\mathrm{Leb}}_{\text{One-dimensional Leb. measure}} \left\{ t \in [0,1] : (t, \psi(t)) \in A \right\} \quad (*)$$

↳ Informally, π is the "diagram" of the permutation ψ.

One can replace $(*)$ with the following equivalent definition indep. of ψ:

Lemma: $\pi([a,b]\times[c,d]) = \mu\left(\eta_1([a,b]) \cap \eta_2([c,d])\right)$, for all rectangles $[a,b]\times[c,d] \subseteq [0,1]^2$.

Proof: By $(*)$

$$\pi([a,b]\times[c,d]) = \mathrm{Leb}\{t\in[a,b] \mid \psi(t)\in[c,d]\}$$

We need to be careful with multiple points of SLEs (but this is OK!)

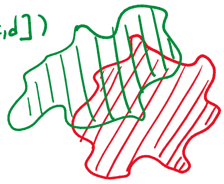
Now, a.s., for almost every $t\in[0,1]$, we have

$$\psi(t)\in[c,d] \iff \eta_2(\psi(t)) \in \eta_2([c,d]).$$

Since $\eta_2(\psi(t)) = \eta_1(t)$ by definition, then

$$\pi([a,b]\times[c,d]) = \mathrm{Leb}\{t\in[a,b] \mid \eta_1(t)\in\eta_2([c,d])\}$$
$$= \mu\left(\eta_1([a,b]) \cap \eta_2([c,d])\right)$$

by our choice of time parametrization. $\square$



Two important special cases:

- In Borga '22, it was shown that for each choice of $(\rho,q)\in(-1,1)\times(0,1)$ for the skew Brownian permuton $\mu_{\rho,q}$, there exists $\gamma\in(0,2)$ and a coupling of two whole-plane SLE_{16/γ^2} curves such that the permuton constructed above coincides with $\mu_{\rho,q}$.

ONLY FOR EXPERTS: (Given $\gamma\in(0,2)$, the parameter $q\in(0,1)$ determines (In a non-explicit way!) an angle $\theta\in(0,\pi)$ so that the two SLEs are coupled with the "Imaginary geometry coupling" (Miller/Sheffield) of angle θ.)

- Togheter with Gwynne and Sun (2022), we conjectured (See our paper for further explanations.) that the permuton limit of meandric permutations is the permuton constructed above when

$$\gamma = \sqrt{\tfrac{1}{3}\left(17-\sqrt{145}\right)}, \quad \kappa_1=\kappa_2=8, \quad \eta_1 \text{ and } \eta_2 \text{ are independent}$$

and so we called it the MEANDRIC PERMUTON.

[Moreover, the map determined by a meander should converge to γ-LQG + 2 ⫫ SLE_8 and the letters should be "determined" by the MEANDRIC PERMUTON (work-in-progress with E. Gwynne)]

SKEW BROWNIAN PERMUTONS

SBP (γ, θ)

- η_1 and η_2 are (strongly) coupled (θ) — one determines the other one.
- $\gamma \in (0,2)$ & $\kappa_1 = \kappa_2 = 16/\gamma^2$
  (MATCHED MODELS $\kappa = 16/\gamma^2$)
- UNIVERSAL FAMILY of PERMUTONS
  (limits of family of permutations)
- 2D-Brownian motion construction
  (Scaling limits are doable (NOT easy))

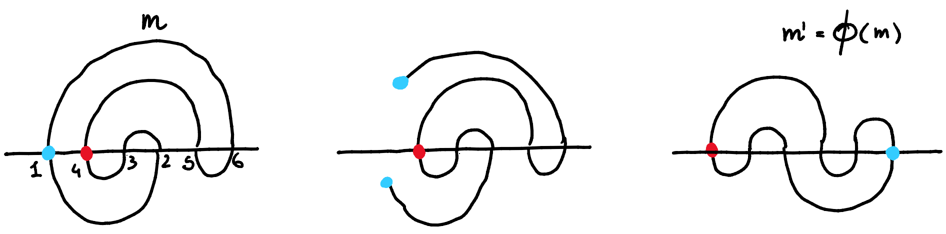
PERMUTONS from INDEPENDENT SLEs

IP $(\gamma, \kappa_1, \kappa_2)$

- η_1 and η_2 are independent
- $\kappa_1, \kappa_2, \gamma \in (4,\infty)^2 \times (0,2)$
- Meandric permuton = IP $\left(8, 8, \sqrt{\tfrac{1}{3}(17-\sqrt{145})}\right)$
  (MISSMATCHED MODEL $\kappa \neq 16/\gamma^2$)
- No 2D-Brownian motion construction
  (Scaling limits are very difficult)

4.6 - Rerooting invariance for the meandric permuton

We consider the following re-rooting operation on meanders:



1 4 3 2 5 6

Open the meander on the left

Close the meander on the right

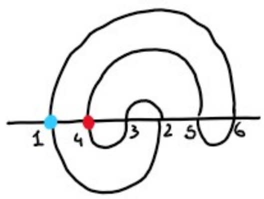
We get a new meander m'

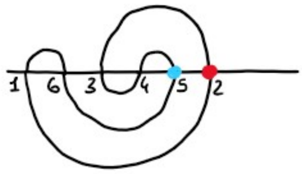
Proposition: If m is a uniform meander then $\phi(m)$ is a uniform meander.

Q: How this property translates in terms of permutations?

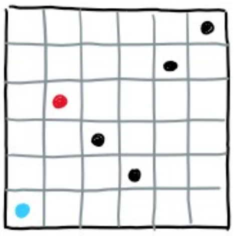
Example: We look at the permutations corresponding to m and $\phi^2(m)$:

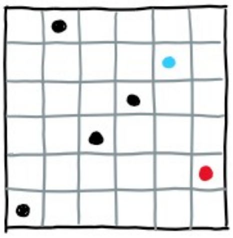
m

$m =$ 

$\phi^2(m) =$ 

1 4 3 2 5 6

$\varphi(m) =$ 

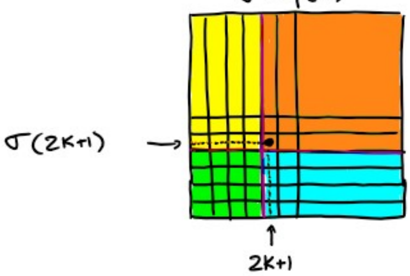
$\varphi(\phi^2(m)) =$ 

To go from $\varphi(m)$ to $\varphi(\phi^2(m))$ we need to do the following operation:

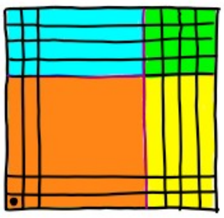
Proposition: Let m be a meander and $k \in \mathbb{Z}_{>0}$.

The diagram of the permutation $\varphi(\phi^{2k}(m))$ is obtained from the diagram of $\varphi(m)$ as follows: divide the diagram of $\sigma = \varphi(m)$ into these four rectangle

$\sigma = \varphi(m)$



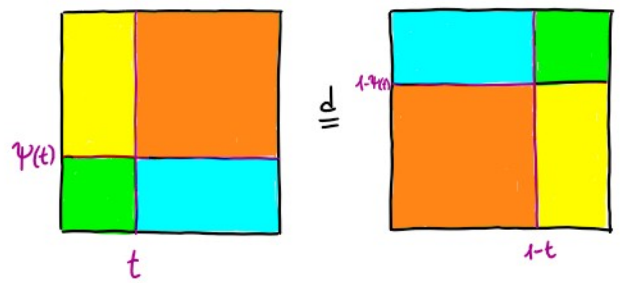
then $\sigma' = \varphi(\phi^{2k}(m))$ has the following diagram:

$\sigma' =$ 

Exercise: Prove the above proposition.

We can prove the same property for the MEANDRIC PERMUTON:

Theorem (B., Guwynne, Son '23) Let π be the meandric permuton. Fix $t \in [0,1]$. The following two permutons are equal in distribution:



where ψ is the function used to define the meandric permuton.

- The proof of the theorem above builds on a certain rerooting invariance property of SLE_8.
- We call the above property "re-rooting invariance for permutons."
- We can prove that $\mathbb{P}(\gamma, 8, 8)$ is rerooting invariant for all $\gamma \in (0,2)$, but we will prove that $\mathbb{P}(\gamma, \kappa_1, \kappa_2)$ is NOT re-rooting invariant as soon as $\kappa_1 \neq 8$ or $\kappa_2 \neq 8$.

4.7 - Some goals for the future

- We would like to find a list of "natural properties" that uniquely characterize the law of the MEANDRIC PERMUTON.
- Re-rooting invariance is obviouly one of these properties
- From the above discussions we now have a convincing explanation on why $\kappa_1 = \kappa_2 = 8$ for the meandric permuton.
- We are working on a "second property" that should be satisfied by $\mathbb{P}(\gamma, \kappa_1, \kappa_2)$ only for a unique value $\gamma^* = \gamma^*(\kappa_1, \kappa_2)$. In particular, we should be able to show that

$$\gamma^*(8,8) = \sqrt{\frac{1}{3}\left(17 - \sqrt{145}\right)}$$

which is exactly the value conjectured for the meandric permuton.

4.8 - Other results on PERMUTONS constructed from SLEs and LQG

In some joint works with Ewain Gwynne, Xin Sun, Nina Holden and Pu Yu, we started to investigate several properties of the permutons π constructed in the previous sections. Here are some results:

(1) We explicitly computed $E[\mu_B]$, when μ_B is the Baxter permuton;

(2) We investigated the behaviours of patterns in the skew Brownian permuton $\mu_{\rho,q}$;

(3) We showed that the Hausdorff dimension of the support of all permutons π constructed in the previous section is always one;

(4) We studied the behaviour of the length of the longest increasing subsequence in permutations converging to the permutons constructed in the previous section.

All these results are obtained combining quite many results/techniques coming from the SLE/LQG literature. We now just focus on one specific result, i.e. (4).

4.9 - The length of the longest increasing subsequence

Definition: For a permutation σ, the length of the longest increasing subsequence $LIS(\sigma)$ is the maximal cardinality of a subset $L \subseteq [1, |\sigma|] \cap \mathbb{Z}$ s.t. the restriction of σ to L is monotone increasing.

THEOREM: (B., Gwynne, Sun, '22)

Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of random permutations s.t. $|\sigma_n| \longrightarrow \infty$ and whose associated permuton π_n satisfies one of the following conditions:

(a) $\mu_{\sigma_n} \xrightarrow{d} IP(\kappa_1, \kappa_2, \gamma)$

(b) $\mu_{\sigma_n} \xrightarrow{d} SBP(\gamma, \theta)$

Then

$$\frac{LIS(\sigma_n)}{|\sigma_n|} \xrightarrow{P} 0.$$

Comments:

(1) The theorem is saying that $LIS(\sigma_n) = o(|\sigma_n|)$.

(2) There exist families of pattern-avoiding permutations $\mathcal{C}$, s.t. if σ_n is uniform in $\mathcal{C}$ then $LIS(\sigma_n) = \Omega(|\sigma_n|)$.

(3) The theorem holds for Baxter permutations.

(4) The law of the length of the longest increasing subsequence in Baxter permutation is equal to the law of the longest directed path in bipolar orientations. The latter can be interpreted as a model of <u>last passage percolation in planar maps</u>.
The scaling limit of this model should be a sort of <u>directed-LQG metric</u>, which should be the "quantum analogue" of the <u>directed landscape</u>.

OPEN PROBLEM: If $\sigma_n \longrightarrow SBP(\gamma, \theta)$, determine $\alpha(\gamma,\theta)$ s.t.

$$|\sigma_n| \sim C \cdot n^{\alpha(\gamma,\theta)}, \quad \text{as} \quad n \to \infty.$$

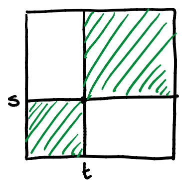
Conjecture:
- $\alpha(\gamma,\theta) = \alpha(\gamma)$, i.e. independent of θ
- $\alpha(\gamma)$ is increasing in γ
- $\alpha(\gamma) \xrightarrow{\gamma \to 0} 1/2$ ⇝ This is the exponent for uniform permutations.
- $\alpha(\gamma) \xrightarrow{\gamma \to 1} 1$

Idea of the proof of the Theorem:

We start with a definition:

Definition: We say that a set $A \subseteq [0,1]^2$ is <u>monotone</u> if

$$A \subseteq [0,t] \times [0,s] \cup [t,1] \times [s,1], \quad \forall (t,s) \in A.$$



Note that for instance the graph of a non-decreasing function (or a subset of it) is monotone.

Proposition 1: Let π be a permuton and let $(\sigma_n)_n$ be a sequence of permutations of size $|\sigma_n| \to \infty$ whose associated permuton $\pi_{\sigma_n} \xrightarrow[n\to\infty]{} \pi$.
Assume that $\pi(A) = 0$ for every monotone set $A \subseteq [0,1]^2$. Then

$$\frac{LIS(\sigma_n)}{|\sigma_n|} \xrightarrow[n\to\infty]{} 0.$$

(The proof uses basic topological facts.)

Proposition 2: Let X be a μ_h-measurable set with the following property:
For μ_h-a.e. pair of points $z, w \in X$, η_1 and η_2 hit z and w in the same order.
Then $\mu_h(X) = 0$.
(The proof is not too long but requires some deeper results for SLEs and LQG)

We now prove the theorem, assuming the two propositions hold.

By assumption, $\mu_{\sigma_n} \xrightarrow{d} \pi \ \big(= IP(K_1, K_2, \gamma) \text{ or } SBP(\gamma, \theta)\big)$
Recall the function $\psi : [0,1] \to [0,1]$ in the definion of π. In particular $\eta_1(t) = \eta_2(\psi(t)), \forall t \in [0,1]$.
For a monotone set $A \subseteq [0,1]^2$, let

$$T_A := \{t \in [0,1] : (t, \psi(t)) \in A\}.$$

Claim: A.s. for every monotone set A, it holds that μ_h-a.e. pair of points $z, w \in \eta_1(T_A)$ are hit in the same order by η_1 and η_2.

Proof: We can assume that η_1 and η_2 have no double points (since a.s. $\mu_h(\text{double points}) = 0$).
Fix $z, w \in \eta_1(T_A)$ and let $t_z, t_w \in [0,1]$ be the times s.t. $\eta_1(t_z) = z$, $\eta_1(t_w) = w$.
By def of T_A, $(t_z, \psi(t_z)) \in A \ni (t_w, \psi(t_w))$. W.l.o.g. we can assume that $t_z < t_w$, and then since A is monotone, we get that $\psi(t_z) < \psi(t_w)$.
By def. of ψ, $\psi(t_z)$ and $\psi(t_w)$ are the times when η_2 hits z and w.
Therefore z and w are hit in the same order by η_1 and η_2. $\square$

Therefore

$$0 \overset{\text{Proposition 2}}{=} \mu_h(\eta_1(T_A)) \overset{\text{By parametrization of } \eta_1}{=} Leb(T_A)$$

Therefore, by definition of π, we get $\pi(A) = Leb(T_A) = 0$.

We proved that $\forall A \subseteq [0,1]^2$ monotone, then $\pi(A) = 0$. By Proposition 1 we can conclude the proof of the theorem. $\square$